DM545/DM871

Linear and Integer Programming

MILP in Spreadsheets

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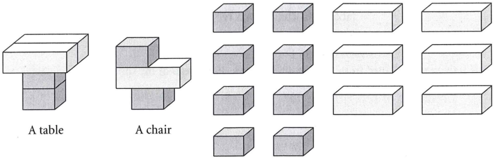
Outline

Outline

Production Planning



	Tables	Chairs	Capacity
Small Pieces	2	2	8
Large Pieces	2	1	6
Profit	16	10	

Mathematical Model:

$$
\begin{aligned}
\max\ 16x_1 + 10x_2 & \\
2x_1 + 2x_2 &\leq 8 \\
2x_1 + x_2 &\leq 6 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}
$$

General Model

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i,\ i = 1, \ldots, m$$
$$x_j \geq 0,\ \ j = 1, \ldots, n$$

$$\begin{aligned} \max\ 16x_1 &+ 10x_2 \\ 2x_1 &+ 2x_2 \leq 8 \\ 2x_1 &+ x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\max\ c^T x$$
$$Ax \leq b$$
$$x \geq 0$$

$$\max \quad \begin{bmatrix} 16\ 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

$$x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

Vector and Matrices in Excel

$$\sum_{j=1}^{n} c_j = c_1 + c_2 + \ldots + c_n$$

$\mathtt{SUM(B5:B14)}$

Scalar product

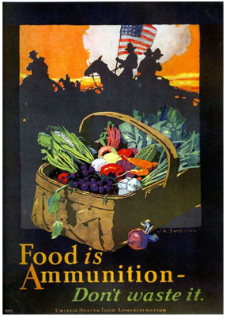
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{j=1}^{n} u_j v_j$$

$\mathtt{SUMPRODUCT(B5:B14, C5:C14)}$

Outline

The Diet Problem (Blending Problems)

- Select a set of foods that will satisfy a set of daily nutritional requirement at minimum cost.
- Motivated in the 1930s and 1940s by US army.
- Formulated as a linear programming problem by George Stigler
- First linear programming problem
- (programming intended as planning not computer code)



min   cost/weight
subject to nutrition requirements:
    eat enough but not too much of Vitamin A
    eat enough but not too much of Sodium
    eat enough but not too much of Calories
    ...

The Diet Problem

Suppose there are:

- 3 foods available, corn, milk, and bread,
- there are restrictions on the number of calories (between 2000 and 2250) and the amount of Vitamin A (between 5000 and 50,000)

Food	Corn	2% Milk	Wheat bread
Vitamin A	107	500	0
Calories	72	121	65
Cost per serving	\$0.18	\$0.23	\$0.05

Rescaled

Food	Corn	2% Milk	Wheat bread
Vitamin A	983.33	2173.91	0
Calories	400	526.08	1300
Cost per serving	\$1	\$1	\$1

The Mathematical Model

$$\min \sum_{i \in F} c_i x_i$$

$$\sum_{i \in F} a_{ij} x_i \geq N_{minj}, \qquad \forall j \in N$$

$$\sum_{i \in F} a_{ij} x_i \leq N_{maxj}, \qquad \forall j \in N$$

$$x_i \geq F_{mini}, \qquad \forall i \in F$$

$$x_i \leq F_{maxi}, \qquad \forall i \in F$$

The History of Stigler's Diet Problem

- The linear program consisted of 9 equations in 77 variables
- Stigler, guessed an optimal solution using a heuristic method
- In 1947, the National Bureau of Standards used the newly developed simplex method to solve Stigler's model.
  It took 9 clerks using hand-operated desk calculators 120 man days to solve for the optimal solution

Outline

Budget Allocation - Exercise 6 of Sheet 3

- A company has six different opportunities to invest money.
- Each opportunity requires a certain investment over a period of 6 years or less.

Expected Investment Cash Flows and Net Present Value									
	Opp. 1	Opp. 2	Opp. 3	Opp. 4	Opp. 5	Opp. 6			Budget
Year 1	-\$5.00	-\$9.00	-\$12.00	-\$7.00	-\$20.00	-\$18.00			\$45.00
Year 2	-\$6.00	-\$6.00	-\$10.00	-\$5.00	\$6.00	-\$15.00			\$30.00
Year 3	-\$16.00	\$6.10	-\$5.00	-\$20.00	\$6.00	-\$10.00			\$20.00
Year 4	\$12.00	\$4.00	-\$5.00	-\$10.00	\$6.00	-\$10.00			\$0.00
Year 5	\$14.00	\$5.00	\$25.00	-\$15.00	\$6.00	\$35.00			\$0.00
Year 6	\$15.00	\$5.00	\$15.00	\$75.00	\$6.00	\$35.00			\$0.00
NPV	\$8.01	\$2.20	\$1.85	\$7.51	\$5.69	\$5.93			

- The company has an investment budget that needs to be met for each year.
- It also has the wish of investing in those opportunities that maximize the combined Net Present Value (NPV) after the 6th year.

Digression: What is the Net Present Value?

- P: value of the original payment presently due
- the debtor wants to delay the payment for t years,
- let r be the market rate of return that the creditor would obtain from a similar investment asset
- the future value of P is $F = P(1+r)^t$

Viceversa, consider the task of finding:

- the present value P of $F = \$100$ that will be received in five years, or equivalently,
- which amount of money today will grow to $F = \$100$ in five years when subject to a constant discount rate? $P = \frac{F}{(1+r)^t}$

Assuming a 5% per year interest rate, it follows that

$$P = \frac{F}{(1+r)^t} = \frac{\$100}{(1+0.05)^5} = \$78.35.$$

Budget Allocation

Net Present Value calculation:
for each opportunity we calculate the NPV at time zero (the time of decision) as:

$$P_0 = \sum_{t=1}^{5} \frac{F_t}{(1+0.05)^5}$$

Expected Investment Cash Flows and Net Present Value									
	Opp. 1	Opp. 2	Opp. 3	Opp. 4	Opp. 5	Opp. 6			Budget
Year 1	-$5.00	-$9.00	-$12.00	-$7.00	-$20.00	-$18.00			$45.00
Year 2	-$6.00	-$6.00	-$10.00	-$5.00	$6.00	-$15.00			$30.00
Year 3	-$16.00	$6.10	-$5.00	-$20.00	$6.00	-$10.00			$20.00
Year 4	$12.00	$4.00	-$5.00	-$10.00	$6.00	-$10.00			$0.00
Year 5	$14.00	$5.00	$25.00	-$15.00	$6.00	$35.00			$0.00
Year 6	$15.00	$5.00	$15.00	$75.00	$6.00	$35.00			$0.00
NPV	$8.01	$2.20	$1.85	$7.51	$5.69	$5.93			

Budget Allocation - Mathematical Model

- Let B_t be the budget available for investments during the years $t = 1..5$.
- Let a_{tj} be the cash flow for opportunity j and c_j its NPV
- Task: choose a set of opportunities such that the budget is never exceeded and the expected return is maximized. Consider both the case of indivisible and divisible opportunities.

Variables $x_j = 1$ if opportunity j is selected and $x_j = 0$ otherwise, $j = 1..6$

Objective

$$\max \sum_{j=1}^{6} c_j x_j$$

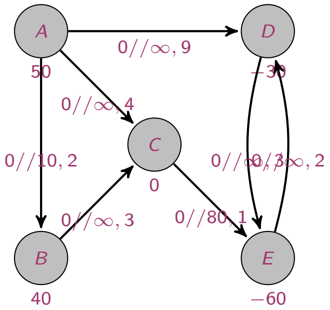
Constraints

$$\sum_{j=1}^{6} a_{tj} x_j + B_t \geq 0 \qquad \forall t = 1..5$$

Outline

Min Cost Flow

A company produces the same product at two different factories, *A* and *B*, and then the product must be shipped to two warehouses, *D* and *E*, where either factory can supply either warehouse. A distribution center *C* can be used to collect the product from *A* and *B* before shipping it to *E*. The distribution network has the following allowed connections: $\{AD, AB, AC, BC, CE, DE, ED\}$ and there are costs and bounds on the amount of product to ship through the connections.



Minimum Cost Network Flows

Find cheapest flow through a network in order to satisfy demands at certain nodes from available supplier nodes.

Variables:

$$x_{ij} \in \mathbb{Z}_0^+$$

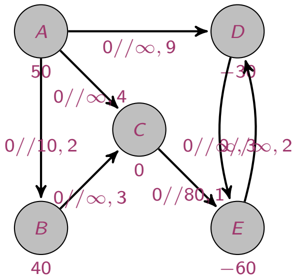
Objective:

$$\min \sum_{ij \in A} c_{ij} x_{ij}$$

Constraints: mass balance + flow bounds

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b(i) \quad \forall i \in V$$

$$l_{ij} \leq x_{ij} \leq u_{ij}$$



Example of mass balance constraints:

$$x_{CE} - x_{AC} - x_{BC} = 0$$

In Matrix Form

	x_{e_1}	x_{e_2}	$\dots$	x_{ij}	$\dots$	x_{e_m}		
	c_{e_1}	c_{e_2}	$\dots$	c_{ij}	$\dots$	c_{e_m}		
1	1	.	$\dots$	.	$\dots$	.	$=$	b_1
2	.	.	$\dots$	.	$\dots$	.	$=$	b_2
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
i	-1	.	$\dots$	1	$\dots$	.	$=$	b_i
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
j	.	.	$\dots$	-1	$\dots$	.	$=$	b_j
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
n	.	.	$\dots$	.	$\dots$	.	$=$	b_n
e_1	1						$\leq$	u_1
e_2		1					$\leq$	u_2
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
(i,j)				1			$\leq$	u_{ij}
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
e_m						1	$\leq$	u_m

$$\min \mathbf{c}^T\mathbf{x}$$
$$N\mathbf{x} = \mathbf{b}$$
$$\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$$

N node arc incidence matrix