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DM545/DM871 – Linear and integer programming

Exercise Sheet 9, Autumn 2026

Exercises with the symbol $^+$ are to be done at home before the class. Exercises with the symbol * will be tackled in class and should be at least read at home. The remaining exercises are left for self training after the exercise class.

Exercise 1$^+$

For each of the following LPs, express the optimal value and the optimal solution in terms of the problem parameters c. If the optimal solution is not unique, it is sufficient to give one optimal solution.

(a)

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{0} \leq \mathbf{x} \leq \mathbf{1} \\ & \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

(b)

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & -\mathbf{1} \leq \mathbf{x} \leq \mathbf{1} \\ & \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

Exercise 2$^+$

In class, we proved that the (minimum) vertex covering problem and the (maximum) matching problem are a weak dual pair. Prove that for bipartite graphs they, actually, are a strong dual pair.

Exercise 3

A managing director has to launch the marketing of a new product. Several candidate products are at his disposal and he has to choose the best one. Hence, he lets each of these products be analysed by a team made of an engineer and a trader who write a review together. The teams are made along the graph in Figure 1; each edge corresponds to a product and its end vertices to the engineer and trader examining it.

a) How many people at least does the managing director gather in order to have the report on all the products? (The report can be given by either the engineer or the trader.)

b) Assuming now that the report must be done jointly by an engineering and a trader, and that each engineer and trader can be occupied with only one candidate product, give a polynomial time algorithm to identify which products will for sure not have the possibility to obtain a report.

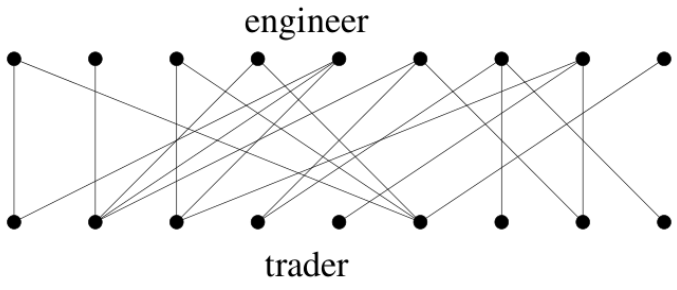


Figure 1:

Exercise 4*

Consider the following three matrices:

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

For each of them say if it is totally unimodular and justify your answer.

Exercise 5*

1. In class we stated that for the uncapaciteted facility location problem there are two formulations:

$$X = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{B}^1 : \sum_{i=1}^{m} x_i \leq my, x_i \leq 1 \text{ for } i = 1, \ldots, m\}$$

$$P = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{R}^1 : x_i \leq y \text{ for } i = 1, \ldots, m, y \leq 1\}$$

   Prove that the polyhedron $P = \{(x_1, \ldots, x_m, y) \in \mathbb{R}^{m+1} : y \leq 1, x_i \leq y \text{ for } i = 1, \ldots, m\}$ has integer vertices. [Hint: start by writing the constraint matrix and show that it is TUM.]

2. Consider the following (integer) linear programming problem:

$$\begin{aligned} \min \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\ & x_3 + x_4 \geq 10 \\ & x_2 + x_3 + x_4 \geq 20 \\ & x_1 + x_2 + x_3 + x_4 \geq 30 \\ & x_2 + x_3 \geq 15 \\ & x_1, x_2, x_3, x_4 \in \mathbb{Z}^+_0 \end{aligned} \tag{1}$$

   The constraint matrix has consecutive 1's in each column. Matrices with consecutive 1's property for each column are totally unimodular. Show that this fact holds for the specific numerical example (1). That is, show first that the constraint matrix of the problem has consecutive 1s in the columns and then that you can transform this matrix into one that you should recognize to be a TUM matrix. [Hint: rewrite the problem in standard form (that is, in equation form) and add a redundant row $\mathbf{0} \cdot \mathbf{x} = 0$ to the set of constraints. Then perform elementary row operations to bring the matrix to a TUM form.]

Exercise 6* MILP Modeling

Shift scheduling. The administrators of a department of a urban hospital have to organize the working shifts of nurses maintaining sufficient staffing to provide satisfactory levels of health care. Staffing requirements at the hospital during the whole day vary from hour to hour and are reported in Table 1.

Hour	Staffing requirement
0 am to 6 am	2
6 am to 8 am	8
8 am to 11 am	5
11 am to 2 pm	7
2 pm to 4 pm	3
4 pm to 6 pm	4
6 pm to 8 pm	6
8 pm to 10 pm	3
10 pm to 12 pm	1

Table 1:

According to union agreements, nurses can work following one of the seven shift patterns in Table 2 each with its own cost.

pattern	Hours of work	total hours	cost
1	0 am to 6 am	6	720 Dkk
2	0 am to 8 am	8	800 Dkk
3	6 am to 2 pm	8	740 Dkk
4	8 am to 4 pm	8	680 Dkk
5	2 pm to 10 pm	8	720 Dkk
6	4 pm to 12 pm	8	780 Dkk
7	6 pm to 12 pm	6	640 Dkk

Table 2:

The department administrators would like to identify the assignment of nurses to working shifts that meets the staffing requirements and minimizes the total cost.

1. Model the problem as a MILP problem.
2. Use the results from the previous exercise to show that the *shift scheduling problem* can be solved efficiently when formulated as a mathematical programming problem. (You do not need to find numerical results.)

Exercise 7*

Generalized Assignment Problem. Suppose there are n types of trucks available to deliver products to m clients. The cost of truck of type i serving client j is c_{ij}. The capacity of truck type i is Q_i and the demand of each client is d_j. There are a_i trucks for each type. Formulate an IP model to decide how many trucks of each type are needed to satisfy all clients so that the total cost of doing the deliveries is minimized. If all the input data will be integer, will the solution to the linear programming relaxation always be integer?

Exercise 8* Network Flows: Problem of Representatives

A town has r residents $R_1, R_2, \ldots, R_r$; q clubs $C_1, C_2, \ldots, C_q$; and p political parties $P_1, P_2, \ldots, P_p$. Each resident is a member of at least one club and belongs to exactly one political party. Each club must nominate one of its members to represent it on the town's governing council so that the number of council members belonging to the political party P_k is at most u_k. Is it possible to find a council that satisfies this "balancing" property?
Show how to formulate this problem as a maximum flow problem.