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DM545/DM871 – Linear and integer programming

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Exercises with the symbol $^+$ are to be done at home before the class. Exercises with the symbol * will be tackled in class and should be at least read at home. The remaining exercises are left for self training after the exercise class.

Exercise 1$^+$

Show that the dual of $\max\{c^T x | Ax = b, x \geq 0\}$ is $\min\{y^T b | y^T A \geq c\}$. Use one of the methods presented in class or even all of them.

Exercise 2

Consider the following LP problem:

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ & 2x_1 + 3x_2 \leq 30 \\ & x_1 + 2x_2 \geq 10 \\ & x_1 - x_2 \leq 1 \\ & x_2 - x_1 \leq 1 \\ & x_1 \geq 0 \end{aligned}$$

- Write the dual problem
- Using the optimality conditions derived from the theory of duality, and without using the simplex method, find the optimal solution of the dual knowing that the optimal solution of the primal is $(27/5, 32/5)$.

Exercise 3*

Consider the problem

$$\begin{aligned} \text{maximize}\quad & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to}\quad & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Without applying the simplex method, how can you tell whether the solution $(2, 0, 1)$ is an optimal solution? Is it? [Hint: consider consequences of Complementary slackness theorem.]

Exercise 4

Consider the following LP:

$$\begin{aligned} \min\ & 3x_1 + 2x_2 - 4x_3 \\ & 2x_1 + x_2 + x_3 \geq 3 \\ & x_1 + x_2 + 2x_3 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Find the optimal solution knowing that the solution of the dual problem is $(u_1, u_2) = (10/3, 11/3)$.

Exercise 5$^+$ LP modeling — Investment plan

An investor has 10,000 Dkk to invest in four projects. The following table gives the cash flow for the four investments.

Project	Year 1	Year 2	Year 3	Year 4	Year 5
1	-1.00	0.50	0.30	1.80	1.20
2	-1.00	0.60	0.20	1.50	1.30
3	0.00	-1.00	0.80	1.90	0.80
4	-1.00	0.40	0.60	1.80	0.95

Expected Investment Cash Flows and Net Present Value	Opp. 1	Opp. 2	Opp. 3	Opp. 4	Opp. 5	Opp. 6	Budget
Year 1	-\$5.00	-\$9.00	-\$12.00	-\$7.00	-\$20.00	-\$18.00	\$45.00
Year 2	-\$6.00	-\$6.00	-\$10.00	-\$5.00	\$6.00	-\$15.00	\$30.00
Year 3	-\$16.00	\$6.10	-\$5.00	-\$20.00	\$6.00	-\$10.00	\$20.00
Year 4	\$12.00	\$4.00	-\$5.00	-\$10.00	\$6.00	-\$10.00	\$0.00
Year 5	\$14.00	\$5.00	\$25.00	-\$15.00	\$6.00	\$35.00	\$0.00
Year 6	\$15.00	\$5.00	\$15.00	\$75.00	\$6.00	\$35.00	\$0.00
NPV	\$8.01	\$2.20	\$1.85	\$7.51	\$5.69	\$5.93	

Figure 1:

The information in the table can be interpreted as follows: For project 1, 1.00 Dkk invested at the start of year 1 will yield 0.50 Dkk at the start of year 2, 0.30 Dkk at the start of year 3, 1.80 Dkk at the start of year 4, and 1.20 Dkk at the start of year 5. The remaining entries can be interpreted similarly. The entry 0.00 indicates that no transaction is taking place. The investor has the additional option of investing in a bank account that earns 6.5% annually. All funds accumulated at the end of 1 year can be reinvested in the following year. Formulate the problem as a linear program to determine the optimal allocation of funds to investment opportunities.
[Taken from Operations Research: An Introduction, Taha]

Exercise 6* LP modeling — Budget Allocation

A company has six different opportunities to invest money. Each opportunity requires a certain investment over a period of 6 years or less. See Figure 1.
The company wants to invest in those opportunities that maximize the combined *Net Present Value* (NPV). It also has an investment budget that needs to be met for each year. (The Net Present Value is calculated with an interest rate of 5%).
How should the company invest?
We assume that it is possible to invest partially in an opportunity. For instance, if the company decides to invest 50% of the required amount in an opportunity, the return will also be 50%.

Net present value:
A debtor wants to delay the payment back of a loan for t years. Let P be the value of the original payment presently due. Let r be the market rate of return on a similar investment asset. The future value of P is

$$F = P(1 + r)^t$$

Viceversa, consider the task of finding the present value P of \$100 that will be received in five years, or equivalently, which amount of money today will grow to \$100 in five years when subject to a constant discount rate. Assuming a 5% per year interest rate, it follows that

$$P = \frac{F}{(1+r)^t} = \frac{\$100}{(1+0.05)^5} = \$78.35.$$

Exercise 7

Consider the following problem:

$$\begin{aligned} \text{maximize} \quad & z = x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

In the ordinary simplex method this problem does not have an initial feasible basis. Hence, the method has to be enhanced by a preliminary phase to attain a feasible basis. Traditionally we talk about a *phase I–phase II* simplex method. In phase I an initial feasible solution is sought and in phase II the ordinary simplex is started from the initial feasible solution found.
There are two ways to carry out phase I.

- Solving an an auxiliary LP problem defined by introducing auxiliary variables and minimizing them in the objective. The solution of the auxiliary LP problem gives an initial feasible basis or a proof of infeasibility.
- Applying the dual simplex on a possibly modified problem to find a feasible solution. If the initial infeasible tableau of the original problem is not optimal then the objective function can be temporarily modified for this phase in order to make the initial tableau optimal although not feasible. Opposite to the primal simplex method, the dual simplex method iterates through infeasible basic solutions, while maintaining them optimal, and stops when a feasible solution is reached.

Dual Simplex: The strong duality theorem states that we can solve the primal problem by solving its dual. You can verify that applying the *primal simplex method* to the dual problem corresponds to the following method, called *dual simplex method* that works on the primal problem:

1. (Feasibility condition) select the leaving variable by picking the basic variable whose right-hand side term is negative, i.e., select i^* with $b_{i^*} < 0$.
2. (Optimality condition) pick the entering variable by scanning across the selected row and comparing ratios of the coefficients in this row to the corresponding coefficients in the objective row, looking for the largest negated. Formally, select j^* such that $j^* = \min\{|c_j/a_{i^*j}| \ : \ a_{i^*j} < 0\}$
3. Update the tableau around the pivot in the same way as with the primal simplex.
4. Stop if no right-hand side term is negative.

Duality can help us with the issue of initial feasible basic solutions. In the problem above, the initial tableau is infeasible and not optimal, hence we cannot apply the primal simplex nor the dual simplex. However, if the objective function was $w = -x_1 - x_2$, then we would have th econditions of infeasibility and optimality needed by the dual simplex. You can understand this also looking at the dual problem. With the new objective function the initial basic solution of the dual problem would be feasible and we could solve the problem solving the dual problem with the primal simplex. In contrast, with original objective function z the primal simplex has infeasible initial basis in both problems. So we can change temporarily the objective function z with w and apply the dual simplex method to the primal problem. When it stops we reached a feasible solution that is optimal with respect to w. We can then reintroduce the original objective function and continue iterating with the primal simplex. The phase I–phase II simplex method that uses the dual simplex is also called the *dual-primal simplex method*.
Apply the two versions of the phase I–phase II simplex method (that is, phase I is carried out with the auxiliary problem or with the dual simplex) to the problem above and verify that they lead to the same solutions.

Exercise 8*

Write the dual of the following problem

$$
\begin{array}{llr}
(P) & \max \sum_{j\in J}\sum_{i\in I} r_j x_{ij} & \\
& \sum_{j\in J} x_{ij} \leq b_i & \forall i \in I \\
& \sum_{i\in I} x_{ij} \leq d_j & \forall j \in J \\
& \sum_{i\in I} p_i x_{ij} = p_j \sum_{i\in I} x_{ij} & \forall j \in J \\
& x_{ij} \geq 0 & \forall i \in I, j \in J
\end{array}
$$