

DM545/DM871 – Linear and integer programming

Exercise Sheet 1, Autumn 2026

Solution:

Included.

Exercises with the symbol $+$ are to be done at home before the class. Exercises with the symbol $*$ will be tackled in class and should be at least read at home. The remaining exercises are left for self training after the exercise class.

Exercises on Modeling Systems of Linear Equations

Exercise 1⁺

A total of \$6,300 was invested in two accounts. Part was invested in obligations at 0.045 annual interest rate and part was invested in a money market fund at 0.0375 annual interest rate. If the total simple interest for one year was \$267.75, then how much was invested in each account?

Solution:

Let x represent the amount invested at 4 and 1/2% annual interest, that is, at 0.045. Let y represent the amount invested at 3 and 3/4% annual interest, that is, at 0.0375. The total amount in both accounts can be expressed as

$$x + y = 6,300$$

To set up a second equation, use the fact that the total interest was \$267.75. The interest for one year is the interest rate times the principal. Use this to add the interest in both accounts.

$$0.045x + 0.0375y = 267.75$$

These two equations together form a linear system.

Solving we obtain that 4,200 was invested at 4 and 1/2% and 2,100 was invested at 3 and 3/4%.

Exercise 2⁺

A construction company produces five different products: p_1, p_2, p_3, p_4, p_5 , using five resources: metal, concrete, plastic, water, electricity.

The amount of resources consumed for producing one unit of products is described by the following python dictionaries:

```
p_1={"metal":0, "concrete":1.3, "plastic":0.2, "water":.8, "electricity":.4}
p_2={"metal":0, "concrete":0, "plastic":1.5, "water":.4, "electricity":.3}
p_3={"metal":.25, "concrete":0, "plastic":0, "water":.2, "electricity":.7}
p_4={"metal":0, "concrete":0, "plastic":.3, "water":.7, "electricity":.5}
p_5={"metal":1.5, "concrete":0, "plastic":.5, "water":.4, "electricity":.8}
```

How much metal is consumed if the company decides to produce the following quantities for each product: 10, 9, 12, 13, 11?

Express the calculation asked by the exercise as a linear algebra operation involving matrices and vectors. Then calculate the numerical answer using Python and numpy. (For example, you can transform the dictionary above into a numpy array as follows: `a=np.array(list(p_1.values()))`.)

Solution:

We can find the answer by scalar product between the row “material” and the vector of production. More precisely, let’s indicate with $\mathbf{p}$ the production vector, with A the matrix of technological coefficients with materials on the rows and products on the columns, with $\mathbf{a}_{.j}$ a column j of A and with $\mathbf{a}_{i.}$ a column i of A . Hence, the answer is given by $\mathbf{a}_{.1} \cdot \mathbf{p}$.

```
###
import numpy as np
p_1={"metal":0, "concrete":1.3, "plastic":0.2, "water":.8, "electricity":.4}
p_2={"metal":0, "concrete":0, "plastic":1.5, "water":.4, "electricity":.3}
p_3={"metal":.25, "concrete":0, "plastic":0, "water":.2, "electricity":.7}
p_4={"metal":0, "concrete":0, "plastic":.3, "water":.7, "electricity":.5}
p_5={"metal":1.5, "concrete":0, "plastic":.5, "water":.4, "electricity":.8}

p = np.array([10,9,12,13,11])
###

A = np.array([list(p_1.values()),
              list(p_2.values()),
              list(p_3.values()),
              list(p_4.values()),
              list(p_5.values())]
             )
np.dot(A.T,p)
```

Exercise 3⁺

Consider again our construction company that produces five different products: p_1, p_2, p_3, p_4, p_5 , using five resources: metal, concrete, plastic, water, electricity.

The amount of resources consumed for producing one unit of products is described by the python dictionaries defined in the previous exercise.

Suppose now that you do not know how much of each product is produced but that you know how much of each material is used. Could you determine from this information the amount produced of each product? Will a solution exist? Will it be unique?

Solution:

The given values are the b_i terms of the system of equalities $A\mathbf{x} = \mathbf{b}$. Hence we can check whether the system is solvable by considerations on the matrix A . We know that a condition for the system to be solvable is that the rank of A is the same as the rank of the augmented matrix of the system. We also know that there will be a unique solution when the matrix A has the same rank as number of variables.

```
%%
np.linalg.matrix_rank(A)
5
```

Hence since the rank of A is 5 and there are 5 materials (the rows of the matrix A) then the augmented matrix has also rank 5 and there the system is consistent. Moreover since there are 5 products then there are no free variables and hence the solution is unique.

Exercises on Modeling (Integer) Linear Programming Problems

The exercises that follow ask you to model the problems in mathematical programming terms. Recall that you have to identify and denote mathematically the:

- i) parameters
- ii) variables (in some cases they might be real and in others integer)

and express as a linear combination of those terms the

- iii) objective function
- iv) constraints.

Remember the quantifiers of the constraints and the domains of the variables.

The description of the model has to be organized in the following parts:

1. **Notation.** Introduction of the mathematical notation: which symbols denote parameters and variables? which sets and indices are you using?
2. **Model.** Mathematical model:

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

(Most likely you will express the model in scalar notation and summations. Remember not to leave any index unspecified.)

3. **Explanation.** Explanation of each line of the mathematical model.

Remember to check that every symbol and index in your model is defined.

See Sec. 1.3.2 from the Lecture Notes [LN] for an example of the expected modeling process and the resulting description.

Note that at the exam your answers must be digitalized, hence it is good to start becoming acquainted with different tools to produce text documents containing mathematical notation and graphs. Read the Tutorial “Preparation for the Written Exam” for a list of useful tools for doing this.

Beside these exercises, you could read Chapter 2 and 3 from [F] and Chapter 2 “Examples” from the book [MG]. You find this book downloadable from SDU library system: <https://www.sdu.dk/da/bibliotek>. Further, the book [Wi] is entirely dedicated to modelling in mathematical programming.

Exercise 4* Brewery Problem

A small brewery produces two types of beer: dark and ale, each requiring different combinations of three limited resources: barley, hops, and yeast.

The recipes for dark and ale require different proportions of resources:

Beverage	Barley (pounds)	Hops (ounces)	Yeast (ounces)	Profit (Dkk)
Dark	15	4	20	23
Ale	5	4	35	13
Quantity	480	160	1190	

The task of the small brewery is to decide the quantity of each type of beer to produce in order to maximize profit from the sale of the two beers, where the profit for each type is given in the table above.

- i. Try to guess the optimal answer to the problem by calculating the total profit of some tentative feasible decisions. For example by producing only one type of beer.
- ii. Model the problem as a linear programming problem and provide the graphical representation of it.
- iii. Later the company decides to introduce a third type of beer: light with the following recipe:

Beverage	Barley (pounds)	Hops (ounces)	Yeast (ounces)	Profit (Dkk)
Light	10	2	15	15

Modify the linear programming model from the previous point in order to consider also this new type of beer.

Solution:

- Devote all resources to ale: 34 barrels of ale $\rightsquigarrow$ 442 Dkk.
- Devote all resources to dark: 32 barrels of beer $\rightsquigarrow$ 736 Dkk.
- 7.5 barrels of ale, 29.5 barrels of beer $\rightsquigarrow$ 776 Dkk.
- 12 barrels of ale, 28 barrels of beer $\rightsquigarrow$ 800 Dkk.

The model is given in Figure 1 and its visualization in 2D in Figure 2. The model for iii) leads to the 3D visualization of Figure 3

	Ale	Beer	
max	13A	+ 23B	Profit
s. t.	5A	+ 15B	≤ 480 Corn
	4A	+ 4B	≤ 160 Hops
	35A	+ 20B	≤ 1190 Malt
	A	, B	≥ 0

Figure 1:

Exercise 5* Food manufacture

A chocolate factory produces two types of chocolate bars: Milk & Hazelnuts (M&H) chocolate and Dark (D) chocolate. The price for a pack of M&H and a pack of D is 100 and 160 Dkk, respectively. Each pack of M&H is made of 50 grams of hazelnuts, 60 grams of chocolate, 40 grams of milk and 50 grams of sugar. Each Dark bar contains 150 grams of chocolate, 50 grams of fat and 30 grams of sugar. The factory has 200 grams of hazelnuts, 1000 grams of chocolate, 250 grams of milk, 300 grams of sugar, and 300 grams of fat left.

Your goal is to determine how many M&H bars and D bars the company should produce to maximize its profit.

- Give the linear program for the problem, using variables x_1 and x_2 and the parameters defined above. Specify (i) the constraints, and (ii) the objective function.
- Graph the feasible region of your linear program. (Tip: look at the Tutorial "Preparation for the Take-Home Exam" in the section "Aiding Tools" in the external web page of the course and find an application that can help you in this task. Those using MacOSx have also the option of the program Grapher).
- Compute the profit at each vertex of the feasible region and report the best solution.

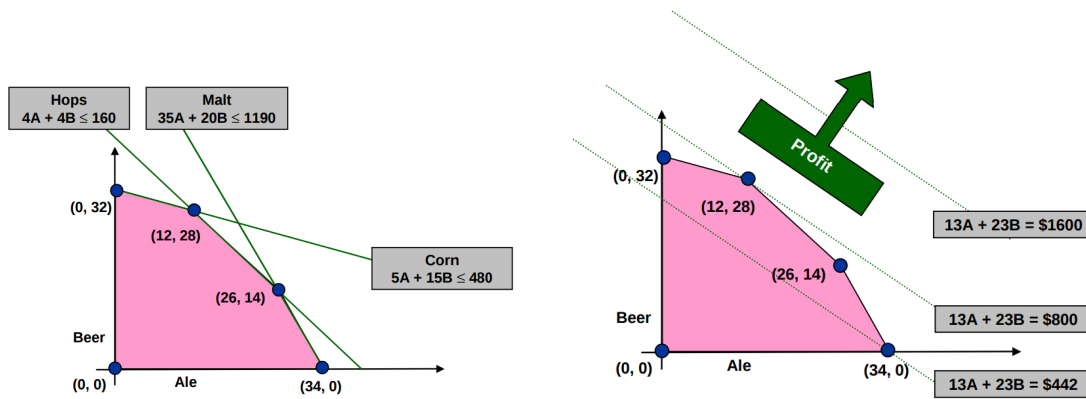


Figure 2:

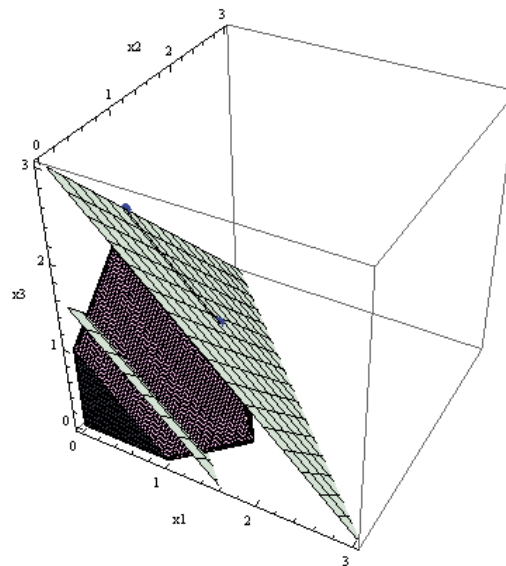


Figure 3:

Solution:

The problem is a factory planning problem. We use the same notation introduced in class.

Notation Let be $J = 1, 2, \dots, n$ indexed by j be the set of products, Milk & Hazelnuts (M&H) chocolate and Dark (D) chocolate, and let $I = 1, 2, \dots, 4$ indexed by i be the set of raw materials (hazelnuts, chocolate, milk, sugar). For each raw material $i \in I$ the factory has b_i units at disposal. To produce a unit of product $j \in J$, a_{ij} units of raw material i are needed. The values a_{ij} are given in the description of the problem and are deterministically known a-priori. Let denote by c_j the profit from selling a unit of product $j \in J$.

To decide the amount of each product to produce, we denote the amount of product $j \in J$ by $x_j \geq 0$. These are the *variables* of our problem. We assume that these numbers are divisible quantities, such that we can accept a result in terms of fractions of a pack.

Model The general (implicit) model is:

$$\max \sum_{j=1}^n c_j x_j \quad (1)$$

$$\sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \quad (2)$$

$$x_j \geq 0, \quad j = 1, \dots, n \quad (3)$$

If we introduce the given data we have the explicit model:

$$\begin{array}{rclcl} \max & 100x_1 & + & 160x_2 & \\ \text{nuts:} & 50x_1 & + & 0x_2 & \leq 200 \\ \text{cacao:} & 60x_1 & + & 150x_2 & \leq 1000 \\ \text{milk:} & 40x_1 & + & 0x_2 & \leq 250 \\ \text{sugar:} & 50x_1 & + & 30x_2 & \leq 300 \\ \text{fat:} & 0x_1 & + & 50x_2 & \leq 300 \\ & x_1 & \geq & 0 & \\ & x_2 & \geq & 0 & \end{array}$$

Explanation The objective function (1) is the total profit from the production expressed as the sum over the products of the profit times the quantity of product produced. The constraints (2) ensure that the raw material needed does not exceed the one available. Finally, the constraints (3) define the domain of the variables.

Each constraint represents an halfspace in 2D defined by the line, whose form is described by the equation of the inequality. For example the first halfplane is the space below the line described by

$$60x_1 + 150x_2 = 1000$$

We can graph all these lines and indicate the feasible region in the Cartesian plane since the problem has only two variables. We can do this by hand or using any available tool for graphing mathematical expressions. From the external web page under of the course under Tools you find the link to LP Grapher, an online app for graphing LP problems. Inputting our data in that app we obtain Figure 4.

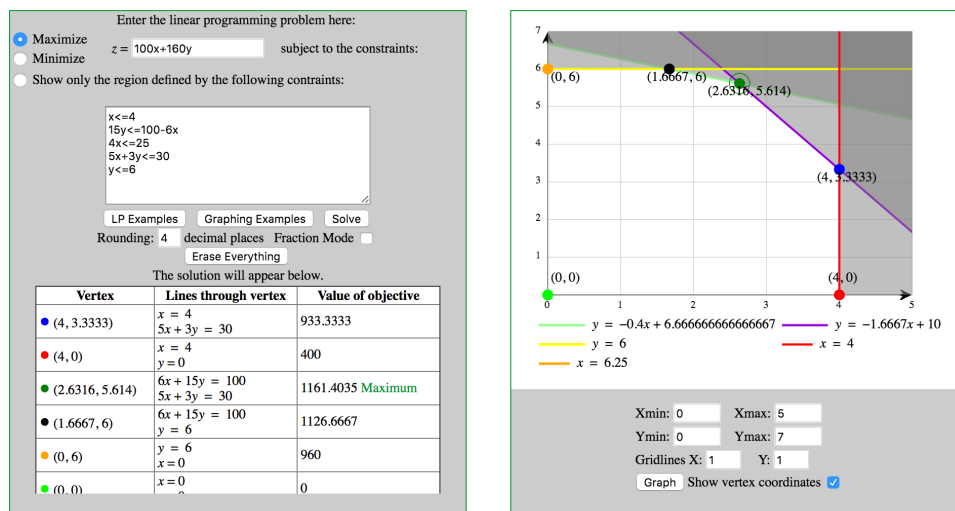


Figure 4: LP grapher for Food Manufacture Exercise.

It would be also possible to plot the objective function. This is the line expressed by the parametric equation:

$$100x_1 + 160x_2 = k$$

where k is a parameter that defines the value of a solution. Since we know that a solution is in a vertex of the polytope, it makes sense to calculate the objective function at each vertex. The vertex where the optimal solution is indicated in the figure.

Exercise 6* Market Survey

An advertising company has to carry out a market survey in order to launch a new product. The survey has to be carried out by telephone, contacting a significant sample of people composed as shown in Table 1:

type of people	married women	unmarried women	married men	unmarried men
number	≥ 150	≥ 110	≥ 120	≥ 100

Table 1:

Calls can be made in the morning (with an operating cost for the company of 1 Euro/call) and in the evening (with an operating cost of 1.6 Euro/call). The average percentage of people reached is as shown in Table 2.

Responder	Morning	Evening
married women	30%	30%
unmarried women	10%	20%
married men	10%	30%
unmarried men	10%	15%
nobody	40%	5%

Table 2:

As it can be seen, evening calls are more expensive, but allow the company to reach a greater number of people: only 5% of the calls is “in vain”. The aim is to minimize the total cost of the calls to be made (morning/evening) so as to reach a significant sample of people. (Problem taken from ref. [F].)

Solution:

Choosing as decision variables

- x_1 , number of calls to be made in the morning
- x_2 , number of calls to be made in the evening

a possible model is the following:

$$\begin{array}{ll}
 \min & 1x_1 + 1.6x_2 & \text{(total cost of the calls)} \\
 \text{s.t.} & 0.3x_1 + 0.3x_2 \geq 150 & \text{(married women)} \\
 & 0.1x_1 + 0.2x_2 \geq 110 & \text{(unmarried women)} \\
 & 0.1x_1 + 0.3x_2 \geq 120 & \text{(married men)} \\
 & 0.1x_1 + 0.15x_2 \geq 100 & \text{(unmarried men)} \\
 & x_1, x_2 \geq 0, \text{ integers} &
 \end{array}$$

In practice the integrality constraints may be removed: if the optimal solution is not integer, it will be sufficient to round up the optimal values x_1 and x_2 , with a negligible increase in the overall cost.

Exercise 7* Optimal Blending

The Metalco Company wants to blend a new alloy (metal) made by 40 percent tin, 35 percent zinc, and 25 percent lead from 5 available alloys having the properties in Table 3.

The objective is to determine the proportions of these alloys that should be blended to produce the new alloy at a minimum cost. Formulate a linear programming model for this problem. [Problem from ref. HL]

To help you in the task we start introducing the mathematical notation that will be used for the model.

Property	Alloy				
	1	2	3	4	5
Percentage of tin	60	25	45	20	50
Percentage of zinc	10	15	45	50	40
Percentage of lead	30	60	10	30	10
Cost (DKK/Kg)	77	70	88	84	94

Table 3:

Let $J = \{1, 2, \dots, 5\}$ indexed by j be the set of alloys and $I = \{\text{tin, zinc, lead}\}$ indexed by i be the set of metals. Let a_{ij} be the fixed parameters that determine the percentage amount of metal i in alloy j . Let c_j be the cost of alloy j in Dkk per Kg. The problem asks to determine the proportions of the alloys to blend to obtain the new alloy with the properties of 40% tin, 35% zinc and 25% lead. Let's call these last parameters $b_i, i \in I$ and the proportion of each alloy to blend with respect to the new alloy by $y_j \geq 0$. Specify the constraints and the objective function using the mathematical terms introduced.

[Units of measure offer a way to test the correctness of the model. In particular, the proportion y_j can be seen as the amount of alloy in Kg per amount of new alloy. Let α be the amount in Kg of the new alloy, and let x_j be the amount in Kg of each alloy from J . Then $y_j = x_j/\alpha$ and it is an adimensional quantity.]

Solution:

If we denote by α the amount in Kg of the new alloy, the blend will satisfy the following equation as far as tin is concerned:

$$\frac{60}{100}x_1 + \frac{25}{100}x_2 + \frac{45}{100}x_3 + \frac{20}{100}x_4 + \frac{50}{100}x_5 = \frac{40}{100}\alpha$$

For the other two components, zinc and lead, the equations will be similar. The value of α is positive and arbitrary. We can divide LHS and RHS by α . Substituting $y_j = x_j/\alpha$ we obtain the amount of alloy per Kg of the new alloy, that is, the proportion we want to determine. We can then find the values of the variables y_j by solving the following LP problem:

$$\min \sum_{j \in J} c_j y_j \quad (4)$$

$$\sum_{j \in J} a_{ij} y_j \geq b_i \quad \forall i \in I \quad (5)$$

$$y_j \geq 0 \quad \forall j \in J \quad (6)$$

The objective function (4) calculates the total cost of the alloy. Note that the total cost would be $c_j x_j$, where c_j is expressed in Dkk/Kg and x_j in Kg. If we use y_j in place of x_j we should interpret the total cost as referring to $\alpha = 1$ Kg of the new alloy. The set of three constraints (5)-(6) express the result of the blend.

Exercise 8

A cargo plane has three compartments for storing cargo: front, center and back. These compartments have capacity limits on both *weight* and *space*, as summarized below:

Compartment	Weight Capacity (Tons)	Space Capacity (Cubic meters)
Front	12	7000
Center	18	9000
Rear	10	5000

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of the weight capacity of that compartment to maintain the balance of the airplane.

Four cargos have been offered for shipment on an upcoming flight as space is available. They are described in Table 4.

Cargo	Weight(Tons)	Volume (Cubic meters/Tons)	Profit
1	20	500	320
2	16	700	400
3	25	600	360
4	13	400	290

Table 4:

Any portion of these cargoes can be accepted. The objective is to determine how much (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight. Formulate a linear programming model for this problem. [Problem from ref. HL]

Solution:

Variables We need to decide how much of each of the four cargoes to put in each of the three compartments. Hence let: x_{ij} be the number of tonnes of cargo j ($j = 1, 2, 3, 4$ for C1, C2, C3 and C4 respectively) that is put into compartment i ($i = 1$ for Front, $i = 2$ for Centre and $i = 3$ for Rear) where $x_{ij} \geq 0$, $i = 1, 2, 3$; $j = 1, 2, 3, 4$. Note here that we are explicitly told we can split the cargoes into any proportions (fractions) that we like.

Constraints

- cannot pack more of each of the four cargoes than we have available

$$\begin{aligned}x_{11} + x_{21} + x_{31} &\leq 20 \\x_{12} + x_{22} + x_{32} &\leq 16 \\x_{13} + x_{23} + x_{33} &\leq 25 \\x_{14} + x_{24} + x_{34} &\leq 13\end{aligned}$$

- the weight capacity of each compartment must be respected

$$\begin{aligned}x_{11} + x_{12} + x_{13} + x_{14} &\leq 12 \\x_{21} + x_{22} + x_{23} + x_{24} &\leq 18 \\x_{31} + x_{32} + x_{33} + x_{34} &\leq 10\end{aligned}$$

- the volume (space) capacity of each compartment must be respected

$$\begin{aligned}500x_{11} + 700x_{12} + 600x_{13} + 400x_{14} &\leq 7000 \\500x_{21} + 700x_{22} + 600x_{23} + 400x_{24} &\leq 9000 \\500x_{31} + 700x_{32} + 600x_{33} + 400x_{34} &\leq 5000\end{aligned}$$

- the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the plane

$$\begin{aligned}x_{11} + x_{12} + x_{13} + x_{14} &= \frac{12}{40} \cdot (x_{11} + x_{12} + x_{13} + x_{14} + x_{21} + x_{22} + x_{23} + x_{24} + x_{31} + x_{32} + x_{33} + x_{34}) \\x_{21} + x_{22} + x_{23} + x_{24} &= \frac{18}{40} \cdot (x_{11} + x_{12} + x_{13} + x_{14} + x_{21} + x_{22} + x_{23} + x_{24} + x_{31} + x_{32} + x_{33} + x_{34}) \\x_{31} + x_{32} + x_{33} + x_{34} &= \frac{10}{40} \cdot (x_{11} + x_{12} + x_{13} + x_{14} + x_{21} + x_{22} + x_{23} + x_{24} + x_{31} + x_{32} + x_{33} + x_{34})\end{aligned}$$

Objective The objective is to maximise total profit, i.e.

$$\max 320/20[x_{11} + x_{21} + x_{31}] + 400/16[x_{12} + x_{22} + x_{32}] + 360/25[x_{13} + x_{23} + x_{33}] + 290/13[x_{14} + x_{24} + x_{34}]$$

In the following we present an alternative solution using a general model.

Let $I = \{1, 2, 3\}$ be the set of compartments in the plane and $J = \{1, 2, 3, 4\}$ the set of cargos. Let's denote by w_j and v_j the weight and the volume of cargo $j \in J$, respectively and by W_i and V_i the capacities of the compartments. Let also p_j be the profit per ton of cargo j . This quantity is obtained by dividing the profit of the whole cargo by its weight. For example, for cargo 1, $p_1 = 320/20$. Finally, let ξ_i be the proportion of a compartment's weight capacity with respect to the total capacity of the airplane:

$$\xi_i = \frac{W_i}{\sum_{i \in I} W_i} \quad \forall i \in I$$

The task is to decide the proportion of each cargo to load in each compartment. Hence, we introduce a variable $x_{i,j}$ for each compartment $i \in I$ and each cargo $j \in J$ representing the proportion of the cargo j that goes in compartment i . Since $x_{i,j}$ is a proportion its value has to be in the closed interval $[0, 1]$. We will also need an auxiliary variable y_i to represent the proportion of weight put in compartment $i \in I$. The model is:

$$\max \sum_{j \in J} p_j \sum_{i \in I} w_j x_{ij} \quad (7)$$

$$\sum_{j \in J} w_j x_{ij} \leq W_i \quad \forall i \in I \quad (8)$$

$$\sum_{j \in J} v_j w_j x_{ij} \leq V_i \quad \forall i \in I \quad (9)$$

$$\sum_{j \in J} x_{ij} = y_i \quad \forall i \in I \quad (10)$$

$$\xi_i \sum_{i \in I} y_i = y_i \quad \forall i \in I \quad (11)$$

$$\sum_i x_{ij} \leq 1 \quad \forall j \in J \quad (12)$$

$$y_i \geq 0 \quad \forall i \in I \quad (13)$$

$$x_{ij} \geq 0 \quad \forall j \in J, \forall i \in I \quad (14)$$

$$x_{ij} \leq 1 \quad \forall j \in J, \forall i \in I \quad (15)$$

The objective function (7) maximizes the total profit. Constraints (8) and (9) ensure that the capacity limits for each compartment are not exceeded. Constraints (10) make the link between the variables x_{ij} and the variables y_i . Constraints (11) impose that the proportions of cargos in the compartment match the proportion of capacities. Finally, constraints (13), (14), (15) define the domain of the variables.

Exercise 9

A small airline flies between three cities: Copenhagen, Aarhus, and Odense. They offer several flights but, for this problem, let us focus on the Friday afternoon flight that departs from Copenhagen, stops in Odense, and continues to Aarhus. There are three types of passengers:

- (a) Those traveling from Copenhagen to Odense.
- (b) Those traveling from Odense to Aarhus.
- (c) Those traveling from Copenhagen to Aarhus.

The aircraft is a small commuter plane that seats 30 passengers. The airline offers three fare classes:

- (a) Y class: full coach.
- (b) B class: nonrefundable.
- (c) M class: nonrefundable, 3-week advanced purchase.

Ticket prices, which are largely determined by external influences (i.e., train and bus competitors), have been set and advertised as in Table 5:

	Copenhagen–Odense	Odense–Aarhus	Copenhagen–Aarhus
Y	300	160	360
B	220	130	280
M	100	80	140

Table 5: The prices advertised.

	Copenhagen–Odense	Odense–Aarhus	Copenhagen–Aarhus
Y	4	8	3
B	8	13	10
M	22	20	18

Table 6: The demand forecast.

Based on past experience, demand forecasters at the airline have determined the upper bounds on the number of potential customers in each of the 9 possible origin-destination/fare-class combinations reported in Table 6.

The goal is to decide how many tickets from each of the 9 origin/destination/ fare-class combinations to sell. The constraints are that the plane cannot be overbooked on either of the two legs of the flight and that the number of tickets made available cannot exceed the forecast maximum demand. The objective is to maximize the revenue.

Formulate this problem as a linear programming problem.

Solution:

Let $I = \{CO, OD, CA\}$ indexed by i be the set of legs and $J = \{Y, B, M\}$ indexed by j the set of fares. Let p_{ij} the prices of the tickets in the various categories and flights.

The decision variables are x_{ij} for $i \in I$ and $j \in J$ and indicate the number of tickets to sell in each category. This number cannot exceed the upper bounds u_{ij} , $i \in I, j \in J$ represented by the forecast demand.

$$\max \sum_{i \in I} \sum_{j \in J} p_{ij} x_{ij} \quad (16)$$

$$\sum_{j \in J} (x_{CO,j} + x_{CA,j}) \leq 30 \quad (17)$$

$$\sum_{j \in J} (x_{OA,j} + x_{CA,j}) \leq 30 \quad (18)$$

$$x_{ij} \leq u_{ij} \quad \forall i \in I, \forall j \in J \quad (19)$$

$$x_{ij} \geq 0 \quad \forall i \in I, \forall j \in J \quad (20)$$

The first constraints ensures that the flight is not overbooked in the leg C to O, while the second constraint ensures that the flight is not overbooked in the leg O-A.

Exercise 10

In this exercise we study the application of linear programming to an area of statistics, namely, regression.

Consider a set of $m = 9$ measurements: 28, 62, 80, 84, 86, 86, 92, 95, 98. A way to summarize these data is by their mean

$$\bar{x} = \frac{1}{m} \sum_{i=1}^m x_i$$

An alternative way is by the median, ie, the measurement that is worse than half of the other scores and better than the other half.

There is a close connection between these statistics and optimization. Show that the mean is the measure that minimizes the sum of squared deviations between the data points and itself and that the median minimizes the sum of the absolute values of the differences between each data point and itself.

Solution:

We want to solve:

$$\min \|x\mathbf{1} - \mathbf{b}\|_1, x \in \mathbb{R}, \mathbf{b} \in \mathbb{R}^n$$

That is we want to minimize $f(x) = \|x\mathbf{1} - \mathbf{b}\|_1 = \sum_{i=1}^n |x - b_i|$.

1. Think of it as a “distance to all points”

Each term $|x - b_i|$ is the distance from x to one data point b_i . So $f(x)$ is the total distance from x to all the points $b_1, \dots, b_n$ on the real line. The minimizer x^* is the point on the real line that minimizes the total distance to all data points.

2. What happens when you move x ?

If you start moving x to the right, every b_i to the left of you increases its distance by 1 unit per step.

Every b_i to the right of you decreases its distance by 1 unit per step.

So the derivative of $f(x)$ (when defined) is:

$$f'(x) = \#\{b_i < x\} - \#\{b_i > x\}.$$

3. Balance point intuition

When more points are on the left, moving right decreases the sum.

When more points are on the right, moving left decreases the sum.

The only way you can't improve is when at least half the points are on each side – exactly the definition of a median.

4. Even vs odd n

If n is odd: there is a unique balance point, the middle data point.

If n is even: the “balancing region” is the whole interval between the two middle values – anywhere in between gives the same minimal distance.

So the median arises naturally as the point that balances the number of data points on each side, ensuring no direction of movement can reduce the total distance.

Next, we want to solve:

$$\min \|x\mathbf{1} - \mathbf{b}\|_2, x \in \mathbb{R}, \mathbf{b} \in \mathbb{R}^n$$

That is we want to minimize $f(x) = \|x\mathbf{1} - \mathbf{b}\|_2 = \sum_{i=1}^n \sqrt{(x - b_i)^2}$.

Equivalently minimize the squared norm

$$g(x) = \sum_{i=1}^n (x - b_i)^2.$$

Differentiate and set to zero:

$$g'(x) = 2 \sum_{i=1}^n (x - b_i) = 2(nx - \sum_{i=1}^n b_i) = 0$$

so

$$x^* = \frac{1}{n} \sum_{i=1}^n b_i$$

Consider now a set of points on a two-dimensional space $S = (x_1, y_1), \dots, (x_m, y_m)$. The points are measurements of a response variable given some control variable, for example, blood pressure given the weight of a person. The points hint at a linear dependency between the variables representing the two dimensions. We may assume a random fluctuation around the right value and hence the following regression model:

$$y = ax + b + \epsilon$$

Specifically, for our set of points S we have

$$y_i = ax_i + b + \epsilon_i, \quad i = 1, \dots, m$$

We thus want to find a line that best fits the measured points. That is, we wish to determine the (unknown) numbers a and b . There is no unique criterion to formulate the desire that a given line “best fits” the points. The task can be achieved by minimizing, in some sense, the vector ϵ . As for the mean and median, we can consider minimizing either the sum of the squares of the ϵ_i 's or the sum of the absolute values of the ϵ_i 's. These concepts are formalized in measure theory by the so called L^p -norm ($1 \leq p < \infty$). For the vector ϵ :

$$\epsilon_p = \left(\sum_i \epsilon_i^p \right)^{1/p}$$

The method of least squares, which is perhaps the most popular, corresponds to L^2 -norm. The minimization of L^2 -norm for the vector ϵ in the variables a and b has a closed form solution that you may have encountered in the statistics courses. This method needs not always to be the most suitable, however. For instance, if a few exceptional points are measured with very large error, they can influence the resulting line a great deal. Just as the median gives a more robust estimate of a collection of numbers than the means, the L^1 norm is less sensitive to outliers than least square regression is. The problem is to solve the following minimization problem:

$$\operatorname{argmin}_{a,b} \sum_i |\epsilon_i| = \operatorname{argmin}_{a,b} \sum_i |ax_i + b - y_i|$$

Unlike for least square regression, there is no explicit formula for the solution of the L^1 -regression problem. However the problem can be formulated as a linear programming problem. Show how this can be done.

The regression via L^∞ -norm corresponds to solving the problem:

$$\operatorname{argmin}_{a,b} \max_{i=1}^n |ax_i + b - y_i|$$

This problem can also be solved by linear programming. Show how to formulate the problem as a linear program.

Solution:

We consider first the L_1 -norm.

The problem we want to solve is:

$$\min_{a,b} \sum_{i=1}^n |ax_i + b - y_i|$$

There are two ways we can rewrite this problem as an LP problem.

In the first, we introduce n new variables, one for each term in the summation, ie, $z_i \in \mathbb{R}$, $i = 1..n$ and write:

$$\begin{aligned} \min \quad & \sum_{i=1}^n z_i \\ \text{s.t.} \quad & z_i \geq ax_i + b - y_i \quad i = 1..n \\ & z_i \geq -(ax_i + b - y_i) \quad i = 1..n \\ & z_i \in \mathbb{R} \quad i = 1..n \\ & a, b \in \mathbb{R} \end{aligned}$$

If the i th term is positive then the value of the corresponding z_i variable is determined by the first constraint, if it is negative, then the value is determined by the second constraint. Hence, we obtained an LP model with n additional variables and $2n$ additional constraints.

The second approach consists in introducing two non-negative variables z_i^+ and z_i^- for each i th term. The LP model is then:

$$\begin{aligned} \min \quad & \sum_{i=1}^n (z_i^+ + z_i^-) \\ \text{s.t.} \quad & ax_i + b - y_i = z_i^+ - z_i^- \quad i = 1..n \\ & z_i^+, z_i^- \geq 0 \quad i = 1..n \\ & a, b \in \mathbb{R} \end{aligned}$$

Since both terms on the right hand side of the first constraint are nonnegative, if the left hand side is positive then z_i^+ will take all the value (any other arrangement would give an higher corresponding value in the objective function for the i th term). If the left hand side is negative then the value will go all in z_i^- . Hence, we obtained an LP problem with $2n$ additional variables and n additional constraints. Empirical experiments can determine which of the two approaches is the best.

For the L_∞ -norm, we want to solve the following problem:

$$\min_{a,b} \left[\max_{i=1..n} \{|ax_i + b - y_i|\} \right]$$

This problem is not in linear form but it can be reduced to a linear form as follows. We introduce one new variable z and write:

$$\begin{array}{ll} \min & z \\ \text{s.t.} & z \geq ax_i + b - y_i \quad i = 1..n \\ & z \geq -(ax_i + b - y_i) \quad i = 1..n \\ & z \in \mathbb{R} \\ & a, b \in \mathbb{R} \end{array}$$

Note that this corresponds to setting

$$z \geq \max\{ax_i + b - y_i, -(ax_i + b - y_i)\}$$

To obtain the LP model we have introduced 1 additional variable and $2n$ constraints.

Exercise 11 Dynamic Input-Output Model

This problem is the dynamic version of [Leontief's Input-Output model](#). A treatment of the model from the point of view of LP is available in Sec. 5.2 of [Wi] (the book is available as ebook from SDU).

An economy consists of three industries: coal, steel and transport. Each unit produced by one of the industries (a unit will be taken as DKK 1's worth of value of production) requires inputs from possibly its own industry as well as other industries. The required inputs and the manpower requirements (also measured in DKK) are given in the Table 7. There is a time lag in the economy so that output in year $t + 1$ requires an input in year t .

Inputs (year t)	Outputs (year $t + 1$), production		
	Coal	Steel	Transport
Coal	0.1	0.5	0.4
Steel	0.1	0.1	0.2
Transport	0.2	0.1	0.2
Manpower	0.6	0.3	0.2

Table 7:

Output from an industry may also be used to build productive capacity for itself or other industries in future years. The inputs required to give unit increases (capacity for DKK 1's worth of extra production) in productive capacity are given in Table 8. Input from an industry in year t results in a (permanent) increase in productive capacity in year $t + 2$.

Inputs (year t)	Outputs (year $t + 2$), productive capacity		
	Coal	Steel	Transport
Coal	0.0	0.7	0.9
Steel	0.1	0.1	0.2
Transport	0.2	0.1	0.2
Manpower	0.4	0.2	0.1

Table 8:

Stocks of goods may be held from year to year. At present (year 0) the stocks and productive capacities (per year) are given in Table 9 (in DKKm). There is a limited yearly manpower capacity of DKK 470m. It is wished to investigate different possible growth patterns for the economy over the next five years. In particular it is desirable to know the growth patterns which would result from pursuing the following objectives:

	Year 0	
	Stocks	Productive capacity
Coal	150	300
Steel	80	350
Transport	100	280

Table 9:

- (i) Maximizing total productive capacity at the end of the five years while meeting an exogenous consumption requirement of DKK 60m of coal DKK 60m of steel and DKK 30m of transport in every year (apart from year 0).
- (ii) Maximizing total production (rather than productive capacity) in the fourth and fifth years, but ignoring exogenous demand in each year.
- (iii) Maximizing the total manpower requirement (ignoring the manpower capacity limitation) over the period while meeting the yearly exogenous demands of (i).

Solution:

See sec. 13.9 of [Wi] (accessible online from within SDU)

The variables in the problem are:

x_{it} total output of industry $i \in \{C, S, T\}$ (C =coal, S =steel, T =transport) in year t

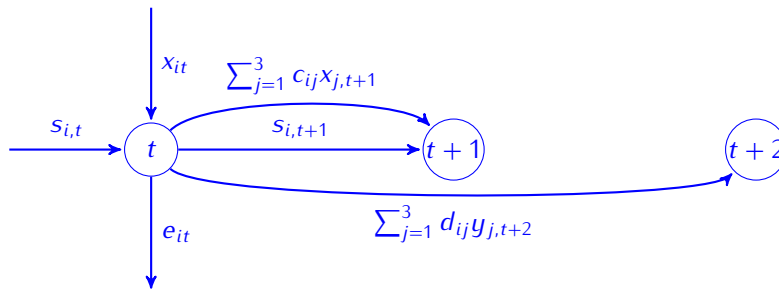
s_{it} stock level of industry $i \in \{C, S, T\}$ at beginning of year t

y_{it} extra productive capacity for industry i becoming effective in year $t = \{2, 3, \dots, 6\}$.

Ignoring exogenous demand, storage and extra productive capacity, at time t the output from the C industry needed to produce C , S and T in time $t + 1$ is given by:

$$x_{coal,t} = 0.1x_{coal,t+1} + 0.5x_{steel,t+1} + 0.4x_{transport,t+1}$$

The exogenous demand for industry i in year t is e_t . Introducing exogenous demand, storage and extra productive capacity we can draw the following mass balance plot at time t for industry i :



Hence at time t we have

$$x_{it} + s_{it} = \sum_{j=1}^3 c_{ij}x_{j,t+1} + \sum_{j=1}^3 d_{ij}y_{j,t+2} + s_{i,t+1} + e_{it}$$

At time t we must ensure that we plan a future need of manpower that does not exceed the capacity:

$$\sum_{i=1}^3 c_{M,i}x_{i,t+1} + \sum_{i=1}^3 d_{M,i}y_{i,t+2} \leq 470$$

Similarly we must ensure that we do not use more than the available productive capacity. Productive capacity added is permanent and hence cumulative through the years (imagine adding plants, infrastructure, etc.).

$$x_{it} \leq y_{i0} + \sum_{l=1}^t y_{il}$$