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# DM545/DM871 – Linear and integer programming

## Exercise Sheet 6, Autumn 2026

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**Solution:**

Included.

Exercises with the symbol $^+$ are to be done at home before the class. Exercises with the symbol $^*$ will be tackled in class and should be at least read at home. The remaining exercises are left for self training after the exercise class.

### Exercise 1* Sensitivity Analysis and Revised Simplex

We reconsider the problem of the furniture-manufacturing company, which we solved in the previous sheet. For completeness we report here again the introductory description.
A furniture-manufacturing company can produce four types of products using three resources.

- A bookcase requires three hours of work, one unit of metal, and four units of wood and it brings in a net profit of 19 Euro.
- A desk requires two hours of work, one unit of metal and three units of wood, and it brings in a net profit of 13 Euro.
- A chair requires one hour of work, one unit of metal and three units of wood and it brings in a net profit of 12 Euro.
- A bedframe requires two hours of work, one unit of metal, and four units of wood and it brings in a net profit of 17 Euro.
- Only 225 hours of labor, 117 units of metal and 420 units of wood are available per day.

In order to decide how much to make of each product so as to maximize the total profit, the managers solve an LP problem.

1) Write the mathematical programming formulation of the problem.

**Solution:**

$$\begin{aligned} \max \quad & 19x_1 + 13x_2 + 12x_3 + 17x_4 \\ & 3x_1 + 2x_2 + x_3 + 2x_4 \leq 225 \\ & x_1 + x_2 + x_3 + x_4 \leq 117 \\ & 4x_1 + 3x_2 + 3x_3 + 4x_4 \leq 420 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

2) With the help of a computational environment such as Python, using only matrix operations, carry out the first iteration of the revised simplex method.

3) determine the value of the solution that has in basis the variables $x_1$, $x_3$ and $x_4$,

4) determine the value of the dual variables associated with the constraints,

5) verify that the solution is indeed optimal.

**Solution:**

The initial tableau is:

```
|-----+-----+-----+-----+-----+-----+-----+----+-----|
| x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | x_7 | -z |   b |
|-----+-----+-----+-----+-----+-----+-----+----+-----|
|   3 |   2 |   1 |   2 |   1 |   0 |   0 |  0 | 225 |
|   1 |   1 |   1 |   1 |   0 |   1 |   0 |  0 | 117 |
|   4 |   3 |   3 |   4 |   0 |   0 |   1 |  0 | 420 |
|  19 |  13 |  12 |  17 |   0 |   0 |   0 |  1 |   0 |
|-----+-----+-----+-----+-----+-----+-----+----+-----|
```

We know that there will be 3 variables in basis. The text of the problem tells us which these 3 variables are: 1, 3, 4. Hence,

$$A_B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 3 & 4 \end{bmatrix} \qquad A_N = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

We can calculate $A_B^{-1}A_N$ in Python or in R:

```
> B=matrix(c(3,1,2,1,1,1,4,3,4),byrow=TRUE,ncol=3)
> B1=solve(B)
> B%*%B1  # check to make sure it is correct!
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
> N=matrix(c( 2, 1, 0, 0, 1, 0, 1, 0, 3, 0, 0, 1),ncol=4,byrow=TRUE)
> B1%*%N
     [,1] [,2] [,3] [,4]
[1,]    1    1    2   -1
[2,]    1    0    4   -1
[3,]   -1   -1   -5    2
> cN=c(13,0,0,0)
> cB=c(19,12,17)
> cN-cB%*%B1%*%N
     [,1] [,2] [,3] [,4]
[1,]   -1   -2   -1   -3
> cB%*%B1%*%c(225,117,420)
> 1827
```

This code gives us:

$$\bar{A} = A_B^{-1}A_N = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 0 & 4 & -1 \\ -1 & -1 & -5 & 2 \end{bmatrix} \qquad x_B^* = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = A_B^{-1}b = \begin{bmatrix} 39 \\ 48 \\ 30 \end{bmatrix}$$

$$\bar{c}_N = \begin{bmatrix} \bar{c}_2 & \bar{c}_5 & \bar{c}_6 & \bar{c}_7 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 & -3 \end{bmatrix}$$

and we can write the final tableau as:

```
|-----+-----+-----+-----+-----+-----+-----+----+-------|
| x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | x_7 | -z |     b |
|-----+-----+-----+-----+-----+-----+-----+----+-------|
|   1 |   1 |   0 |   0 |   1 |   2 |  -1 |  0 |    39 |
|   0 |   1 |   1 |   0 |   0 |   4 |  -1 |  0 |    48 |
|   0 |  -1 |   0 |   1 |  -1 |  -5 |   2 |  0 |    30 |
|   0 |  -1 |   0 |   0 |  -2 |  -1 |  -3 |  1 | -1827 |
|-----+-----+-----+-----+-----+-----+-----+----+-------|
```

Since all reduced costs are negative then the tableau and the corresponding solution are optimal.
Alternative solution in Python. Input parameters not yet updated to the current exercise.

```
# %%
import numpy as np
np.set_printoptions(precision=3,suppress=True)

c = np.array([19, 13, 12, 17])
A = np.array([[3, 2, 1, 2],
              [1, 1, 1, 1],
              [4, 3, 3, 4]])
```

From the matrix interpretation of the simplex we know that the multipliers $\pi$ that bring us to the optimal tableau are given by:

$$\pi = -\mathbf{c}_B A_B^{-1}$$

The matrix $A_B$ for $\mathbf{x}_B = [x_1, x_3, x_4]$ and $A_N$ for $x_N = [x_2, x_5, x_6, x_7]$ are constructed below. From the proof of the strong duality theorem we also know that the dual values $\mathbf{y}$ are the multipliers with a change of sign: $\mathbf{y} = -\pi$. Hence, the dual values are $\mathbf{y} = \mathbf{c}_B A_B^{-1}$. Note that Python is zero-indexed.

```
# %%
import numpy as np
np.set_printoptions(precision=3,suppress=True)

c = np.array([19, 13, 12, 17])
A = np.array([[3, 2, 1, 2],
              [1, 1, 1, 1],
              [4, 3, 3, 4]])
b = np.array([225, 117, 420])

# %%
A = np.concatenate([A,np.identity(3)],axis=1)
c = np.concatenate([c,np.zeros(3)],axis=0)

B = np.array([0,2,3])
mask = np.ones(7, dtype=bool)
mask[B] = False
N = np.arange(7)[mask]

print(N)

# %% dual values
A_B = A[:,B]
c_B = c[B]

A_B_i = np.linalg.inv(A_B)
y = np.dot(c_B,A_B_i)
```

```
[1 4 5 6]
[2. 1. 3.]
```

Hence, [2.1.3.] are the values of the dual variables.

The values of the variables in basis are [39.48.30.] as calculated below:

```
print(y)
# %% primal values
x = A_B_i @ b

print(x)
```

```
[39. 48. 30.]
```

Hence, the whole solution is $[39., 0, 0, 48.30., 0, 0]$.

To verify the optimality we can show that all reduced costs are nonpositive:

```
# %% proof of optimality

A_N = A[:,[0,1,3,5,6]]
c_N = c[[0,1,3,5,6]]

c_bar_N = c_N - np.dot(y,A_N)
print(c_bar_N)


x_B = np.dot(A_B_i,b)
print(x_B)
```

```
[ 0. -1.  0. -1. -3.]
[39. 48. 30.]
```

or calculate the value of the primal and dual solutions and show that they match:

```
# %% proof of optimality

print(c_B @ x)
print(y @ b)
```

```
1827.0
1827.0
```

## Exercise 2 Factory Planning and Machine Maintenance

Tasks 1-3 of Factory Planning and Maintainance Case.

## Exercise 3* Crop Planning

The following are two cases of crop planning that arise in agriculture. They are two separated cases and do not share information. The first case considers the single period scenario while the second a multi-period scenario. In real-life applications, a combination of the two cases is the most likely needed.

### Subtask 3.1

Increasing the efficiency in agriculture by exploring the possibilities of optimal allocation of the scarce resources for the various farms has become a urgent need of society. Crop planning is one of the important stages in crop production and it has a considerable effect on farmer revenue.
Assume we have collected data regarding land allocation, labor days, machine hours, cost of seeds, fertilizers, and pesticides from farmers for four crops: paddy, cotton, chilli and maize, which are grown in a region.
Using averages of these data we want to determine the crop plan that maximizes the net return for an average farmer of a given region. The profit that farmers make per hectare (1 ha =10.000 $m^2$) of arable area of the four crops, paddy, cotton, chilli and maize are in Danish krone: 15.434, 46.186, 160.607, and 14.088, respectively.
The resource constraints to be considered are land, labor in man days, machine hours, operating cost and alternating crop plans.

- Land. Land is the basic resource of production. All land related activities are expressed in hectares basis. The total land available that we consider is 46 hectares and the total arable area cannot exceed this size.

- Labor. The labor is taken in man days. The man days needed per hectare of cultivation of the four crops are 29, 58, 308 and 24, respectively and the capacity is 5819 days.
- Machine hours. Farmers are hiring the tractor services for land preparation and machine hours are limited. The machine hours required per hectare of area of the four crops are 5, 4, 2, and 4, respectively, and the capacity is 525 hours.
- Operating capital. The operating capital refers to the funds required to meet the cost of seeds, fertilizers, farm yard manure, plant protection chemicals, insurance charges, marketing expenses and wages of the human and bullock labor and tractor power. The operating costs per hectare of area of the four crops are 32.474, 38.897, 111.962, 21.713, respectively, and the maximum budget is 4.343.868 Dkk.
- Alternate crop plans. The alternate crop plans are developed by introducing minimum and maximum land allocation constraints for the crops because in practical situations it is not applicable to allocate the whole land to one crop due to the fluctuations in the demand and risk associated with specialization of one crop. Here, we consider only a limitation on the crop that is cultivated by the majority of the farmers, paddy. The portion of (total available) land that can be allocated to paddy crop must be reduced by 20%.

Formulate the problem as a linear programming problem.

**Solution:**

The decision variables of LP model are:

- $x_1$, hectares allocated for paddy crop
- $x_2$, hectares allocated for cotton crop
- $x_3$, hectares allocated for chilli crop
- $x_4$, hectares allocated for maize crop

The LP model is given by

$$\begin{aligned}
\max z = & 15434x_1 + 46186x_2 + 160607x_3 + 14088x_4 \\
\text{s.t. } & x_1 + x_2 + x_3 + x_4 \leq 46 \\
& 29x_1 + 58x_2 + 308x_3 + 24x_4 \leq 5819 \\
& 5x_1 + 4x_2 + 2x_3 + 4x_4 \leq 525 \\
& 32474x_1 + 38897x_2 + 111962x_3 + 21713x_4 \leq 4343868 \\
& x_1 \leq 0.8 \cdot 46 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

The first constraint enforces the land constraint, the second constraint enforces the labor man days, the third enforces the machine hours and the fourth constraint enforces the overall costs to stay below the budget. The fifth constraint tries to ensure alternation in the crop plan. (During the exam an announcement was sent out specifying that "No more than 20% of the (total available) land may be allocated to paddy". In this case the constraint should be: $x_1 \leq 0.2 \cdot 46$. Both versions will be accepted as correct.)

### Subtask 3.2

Common Agricultural Policies stipulate a "payment for agricultural practices which are beneficial for the climate and the environment". This is commonly known as "greening payment" and represents a direct aid per hectare rewarding agricultural sustainable practices.
The greening payment may potentially encourage farmers to meet certain environmental requirements in return for governmental support payments. This greening aid rewards farmers complying three basic "greening rules" (or equivalent practices). These are: (1) maintenance of existing permanent grassland (2) establishment of an ecological focus area on arable land and (3) crop diversification.

We consider a farmer who needs to decide whether to adapt his/her cropping plan in order to fulfill the greening rules and therefore receive the greening subsidy. The farmer seeks to maximize his/her revenues over the time horizon of the policy, that is, the farmer seeks to optimize cropping plan decisions for the next five years by maximizing his/her net return.
Let us consider a farmer that owns a piece of arable land of 35 hectares (ha) where $N = 6$ different crops may be grown. We consider a planning period of $T = 5$ years, with one growing season per year. We are given the crop yields, $e_i$, and the cost of exploitation, $c_i$, of each crop $i = 1..6$. Moreover, the prices $p_i$ received by farmers for each crop and the value of the greening subsidy payment $b_i$ for each crop are available. These data are shown in Figure 1. The first table above shows the crops considered, their selling price and their exploitation cost, the other two tables below show the yield and the basic payment per crop type. All this data is per unit of hectare and we assume a linear growth with respect to cultivated hectares for the corresponding crop.
The price received by the farmer and the exploitation cost of the crops are capitalized each year by recursively updating the current price and cost of the corresponding crop ($p_i$ and $c_i$, respectively) at a given inflation rate (i.e., consumer price index) $r = 5\%$:

$$p_{it} = p_i \cdot (1 + r)^t \quad i = 1..N, t = 1..T$$

$$c_{it} = c_i \cdot (1 + r)^t \quad i = 1..N, t = 1..T$$

The farmer's net return $NV_t$ (€) at the end of period $t$ must be actualized to the *net present value* at $t = 0$. The total net present value accumulated per hectare of each crop $i = 1..N$ during the planning horizon can be derived by means of the following formula:

$$NPV_i = \sum_{t=1}^{T} \frac{(e_i p_{it} + b_i - c_{it})}{(1 + a)^t}$$

where $a$ is a discount rate of 5%.
The goal is to determine the optimal cropping plan by choosing the crops to be grown every year and their hectareage in order to maximize the total farmer's economic net present value over the time horizon of five years.
The greening subsidy $b_i$ is however only received if the three basic greening rules of the policy are fulfilled: (1) maintenance of permanent grassland (2) establishment of ecological focus areas and (3) crop diversification.
More precisely, the restrictions imposed by the greening rule are as follows:

- Arable land. The land allocated to the crops, the permanent grassland area and the ecological focus area must not exceed the total farm arable land each year.
- Maintenance of permanent grassland. The permanent grassland has to be at least 5% of the total arable land every year.
- Establishment of ecological focus areas. The ecological focus area has to be greater than 5% of the total arable land in the previous year up to year $t = 2$, and greater than 7% of the previous year from $t = 3$.
- Crop diversification. The farmer must grow at least three different crops. The main crop (i.e., the crop occupying the largest area) must not cover more than 75% of the total arable land in each period $t$. The two largest crops together must not cover more than 95% of the arable land.
- Crop rotations. The same crop cannot be grown in the same area in sequential seasons. To allow this it must be enforced that the arable land occupied by one crop in period $t$ is lower or equal to the surface not occupied by this crop during the previous period. For the first planning period we assume the area occupied by the crops in the previous year was 0 ha.

A mathematical model of the problem and its solution can provide valuable insight to both farmers and Governments: farmers can optimize their revenue and Governments can determine the minimum subsidy value that would make the implementation of greening rules economically appealing (eg, by comparing farmer's revenue with and without the greening rules).
Formulate the problem as a linear programming problem.

| $i$ | Crop | Price (€t$^{-1}$) | Cost (€ha$^{-1}$) |
|---|---|---|---|
| 1 | Wheat | 230.1 | 604.1 |
| 2 | Barley | 194.8 | 598.1 |
| 3 | Rye | 181.5 | 377.6 |
| 4 | Oat | 183.6 | 585.7 |
| 5 | Dried Pea | 223.6 | 455.6 |
| 6 | Sunflower | 380.1 | 430.9 |

**Yield (t ha$^{-1}$)**

| Wheat | Barley | Rye | Oat | Dried Pea | Sunflower |
|---|---|---|---|---|---|
| 2.396 | 2.334 | 1.468 | 1.642 | 0.811 | 0.774 |

**Basic payment (€ha$^{-1}$)**

| Wheat | Barley | Oat | Rye | Dried peas | Sunflower |
|---|---|---|---|---|---|
| 150.96 | 147.05 | 92.46 | 103.45 | 181.00 | 48.73 |

Figure 1: Above: Crops considered, selling prices and production costs per crops. Below left: Crop yields. Below right: Basic payment for crops.

**Solution:**

Let $S_t$ represent the total surface of the arable land (ha). It is $S_t = 35$ for all $t = 1..T$ and $S_0 = 25$. Let the decision variables be: $x_{it} \geq 0$ to represent the arable area in hectares (ha) to be allocated to crop $i$ in period $t$, $y_t \geq 0$ to represent the permanent grassland area (ha) in period $t$ and $z_t \geq 0$ to represent the ecological focus area (ha) in period $t$.

$$\max \sum_{i=1}^{N} \sum_{t=1}^{T} \frac{(e_i p_{it} + b_i - c_{it}) x_{it}}{(1+a)^t} \tag{1}$$

$$y_t + z_t + \sum_{i=1}^{N} x_{it} \leq S_t \qquad t = 1..T \tag{2}$$

$$y_t \geq 0.05 \cdot S_t \qquad \forall t = 1..T \tag{3}$$

$$z_t \geq 0.05 \cdot S_{t-1} \qquad t = 1, 2 \tag{4}$$

$$z_t \geq 0.07 \cdot S_{t-1} \qquad t = 3, 4, 5 \tag{5}$$

$$x_{it} \leq 0.75 \cdot (S_t - y_t - z_t) \qquad i = 1..N,\ t = 1..T \tag{6}$$

$$x_{it} + x_{jt} \leq 0.95 \cdot (S_t - y_t - z_t) \qquad i, j = 1..N, i \neq j\ t = 1..T \tag{7}$$

$$x_{it} \leq S_t - y_t - z_t - x_{i,t-1} \qquad i = 1..N,\ t = 1..T \tag{8}$$

$$x_{it} \geq 0 \qquad i = 1..N,\ t = 1..T \tag{9}$$

$$y_t \geq 0 \qquad t = 1..T \tag{10}$$

$$z_t \geq 0 \qquad t = 1..T \tag{11}$$

(12)

Constraint (2) expresses the arable land constraint. Constraints (3) ensure the desired minimum grassland. Constraints (4)-(5) ensure the desired minimum focus areas for every year. Constraints (6)-(7) enforce the rotation by limiting the amount of arable land for the largest crops. Constraints (8) makes it feasible to achieve the rotations of the crops.