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DM545/DM871 – Linear and integer programming

Sheet 2, Spring 2024 [pdf format]

Solution:
Included.

Exercise 1

Solve the following LP problem

$$
\begin{aligned}
\text{maximize} \quad & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
\text{subject to} \quad & x_1 \leq 1 \\
& 1/2x_1 - 11/2x_2 - 5/2x_3 + 9x_4 \leq 0 \\
& +1/2x_1 - 3/2x_2 - 1/2x_3 + x_4 \leq 0 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

using the following pivot rule:

i. the entering variable will always be the nonbasic variable that has the largest coefficient in the z-row of the dictionary.

ii. if two or more basic variables compete for leaving the basis, then the candidate with the smallest subscript will be made to leave.

There are a few calculations to carry out, hence you are reccomended to use Python. See the guidelines in the Tutorial: Python for matrix operations.

Solution:

```
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     x7 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|    1/2 |  -11/2 |   -5/2 |      9 |      1 |      0 |      0 |      0 |      0 |
|    1/2 |   -3/2 |   -1/2 |      1 |      0 |      1 |      0 |      0 |      0 |
|      1 |      0 |      0 |      0 |      0 |      0 |      1 |      0 |      1 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|     10 |    -57 |     -9 |    -24 |      0 |      0 |      0 |      1 |      0 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
PRIMAL SIMPLEX

pivot column: 1
pivot row: 1
 pivot: 1/2


|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     x7 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      1 |    -11 |     -5 |     18 |      2 |      0 |      0 |      0 |      0 |
|      0 |      4 |      2 |     -8 |     -1 |      1 |      0 |      0 |      0 |
|      0 |     11 |      5 |    -18 |     -2 |      0 |      1 |      0 |      1 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      0 |     53 |     41 |   -204 |    -20 |      0 |      0 |      1 |      0 |
```

```
|--------+--------+--------+--------+--------+--------+--------+--------+--------+

pivot column: 2
pivot row: 2
 pivot: 4

|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      x1 |      x2 |      x3 |      x4 |      x5 |      x6 |      x7 |      -z |       b |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|       1 |       0 |     1/2 |      -4 |    -3/4 |    11/4 |       0 |       0 |       0 |
|       0 |       1 |     1/2 |      -2 |    -1/4 |     1/4 |       0 |       0 |       0 |
|       0 |       0 |    -1/2 |       4 |     3/4 |   -11/4 |       1 |       0 |       1 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|       0 |       0 |    29/2 |     -98 |   -27/4 |   -53/4 |       0 |       1 |       0 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+

pivot column: 3
pivot row: 1
 pivot: 1/2

|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      x1 |      x2 |      x3 |      x4 |      x5 |      x6 |      x7 |      -z |       b |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|       2 |       0 |       1 |      -8 |    -3/2 |    11/2 |       0 |       0 |       0 |
|      -1 |       1 |       0 |       2 |     1/2 |    -5/2 |       0 |       0 |       0 |
|       1 |       0 |       0 |       0 |       0 |       0 |       1 |       0 |       1 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|     -29 |       0 |       0 |      18 |      15 |     -93 |       0 |       1 |       0 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+

pivot column: 4
pivot row: 2
 pivot: 2

|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      x1 |      x2 |      x3 |      x4 |      x5 |      x6 |      x7 |      -z |       b |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      -2 |       4 |       1 |       0 |     1/2 |    -9/2 |       0 |       0 |       0 |
|    -1/2 |     1/2 |       0 |       1 |     1/4 |    -5/4 |       0 |       0 |       0 |
|       1 |       0 |       0 |       0 |       0 |       0 |       1 |       0 |       1 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|     -20 |      -9 |       0 |       0 |    21/2 |  -141/2 |       0 |       1 |       0 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+

pivot column: 5
pivot row: 1
 pivot: 1/2

|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      x1 |      x2 |      x3 |      x4 |      x5 |      x6 |      x7 |      -z |       b |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      -4 |       8 |       2 |       0 |       1 |      -9 |       0 |       0 |       0 |
|     1/2 |    -3/2 |    -1/2 |       1 |       0 |       1 |       0 |       0 |       0 |
|       1 |       0 |       0 |       0 |       0 |       0 |       1 |       0 |       1 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      22 |     -93 |     -21 |       0 |       0 |      24 |       0 |       1 |       0 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
```

```
pivot column: 6
pivot row: 2
 pivot: 1
```

x1	x2	x3	x4	x5	x6	x7	-z	b
1/2	-11/2	-5/2	9	1	0	0	0	0
1/2	-3/2	-1/2	1	0	1	0	0	0
1	0	0	0	0	0	1	0	1
10	-57	-9	-24	0	0	0	1	0

```
pivot column: 1
pivot row: 1
 pivot: 1/2
```

Thus we discover that we return to the first tableau and that therefore we are cycling. We are in a malignous degenerancy. In order to make it benignous, that is, in order to avoid cycling a different pivoting rule must be used. The sign that we are in a degenerate case that might turn out malignous is the fact that one of the b_i terms is zero. This implies that there is a basic variable that gets value zero.

Exercise 2

Solve the following problem, known as the Klee-Minty problem, using the largest coefficient pivoting rule.

$$\begin{aligned} \text{maximize} \quad & 100x_1 + 10x_2 + x_3 \\ \text{subject to} \quad & x_1 \leq 1 \\ & 20x_1 + x_2 \leq 100 \\ & 200x_1 + 20x_2 + x_3 \leq 10000 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Can you generalize the example to n variables and guess what will be the number of iterations the simplex will do?

Solution:

x1	x2	x3	x4	x5	x6	-z	b
1	0	0	1	0	0	0	1
20	1	0	0	1	0	0	100
200	20	1	0	0	1	0	10000
100	10	1	0	0	0	1	0

```
PRIMAL SIMPLEX

pivot column: 1
pivot row: 1
 pivot: 1
1
```

x1	x2	x3	x4	x5	x6	-z	b
1	0	0	1	0	0	0	1

```
|        0 |        1 |        0 |      -20 |        1 |        0 |        0 |       80 |
|        0 |       20 |        1 |     -200 |        0 |        1 |        0 |     9800 |
|---------+---------+---------+---------+---------+---------+---------+---------+
|        0 |       10 |        1 |     -100 |        0 |        0 |        1 |     -100 |
|---------+---------+---------+---------+---------+---------+---------+---------+

pivot column: 2
pivot row: 2
 pivot: 1
1

|---------+---------+---------+---------+---------+---------+---------+---------+
|       x1 |       x2 |       x3 |       x4 |       x5 |       x6 |       -z |        b |
|---------+---------+---------+---------+---------+---------+---------+---------+
|        1 |        0 |        0 |        1 |        0 |        0 |        0 |        1 |
|        0 |        1 |        0 |      -20 |        1 |        0 |        0 |       80 |
|        0 |        0 |        1 |      200 |      -20 |        1 |        0 |     8200 |
|---------+---------+---------+---------+---------+---------+---------+---------+
|        0 |        0 |        1 |      100 |      -10 |        0 |        1 |     -900 |
|---------+---------+---------+---------+---------+---------+---------+---------+

pivot column: 4
pivot row: 1
 pivot: 1
1

|---------+---------+---------+---------+---------+---------+---------+---------+
|       x1 |       x2 |       x3 |       x4 |       x5 |       x6 |       -z |        b |
|---------+---------+---------+---------+---------+---------+---------+---------+
|        1 |        0 |        0 |        1 |        0 |        0 |        0 |        1 |
|       20 |        1 |        0 |        0 |        1 |        0 |        0 |      100 |
|     -200 |        0 |        1 |        0 |      -20 |        1 |        0 |     8000 |
|---------+---------+---------+---------+---------+---------+---------+---------+
|     -100 |        0 |        1 |        0 |      -10 |        0 |        1 |    -1000 |
|---------+---------+---------+---------+---------+---------+---------+---------+

pivot column: 3
pivot row: 3
 pivot: 1
1

|---------+---------+---------+---------+---------+---------+---------+---------+
|       x1 |       x2 |       x3 |       x4 |       x5 |       x6 |       -z |        b |
|---------+---------+---------+---------+---------+---------+---------+---------+
|        1 |        0 |        0 |        1 |        0 |        0 |        0 |        1 |
|       20 |        1 |        0 |        0 |        1 |        0 |        0 |      100 |
|     -200 |        0 |        1 |        0 |      -20 |        1 |        0 |     8000 |
|---------+---------+---------+---------+---------+---------+---------+---------+
|      100 |        0 |        0 |        0 |       10 |       -1 |        1 |    -9000 |
|---------+---------+---------+---------+---------+---------+---------+---------+

pivot column: 1
pivot row: 1
 pivot: 1
1

|---------+---------+---------+---------+---------+---------+---------+---------+
|       x1 |       x2 |       x3 |       x4 |       x5 |       x6 |       -z |        b |
```

```
|--------+--------+--------+--------+--------+--------+--------+--------+
|      1 |      0 |      0 |      1 |      0 |      0 |      0 |      1 |
|      0 |      1 |      0 |    -20 |      1 |      0 |      0 |     80 |
|      0 |      0 |      1 |    200 |    -20 |      1 |      0 |   8200 |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      0 |      0 |      0 |   -100 |     10 |     -1 |      1 |  -9100 |
|--------+--------+--------+--------+--------+--------+--------+--------+

pivot column: 5
pivot row: 2
 pivot: 1
1

|--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      1 |      0 |      0 |      1 |      0 |      0 |      0 |      1 |
|      0 |      1 |      0 |    -20 |      1 |      0 |      0 |     80 |
|      0 |     20 |      1 |   -200 |      0 |      1 |      0 |   9800 |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      0 |    -10 |      0 |    100 |      0 |     -1 |      1 |  -9900 |
|--------+--------+--------+--------+--------+--------+--------+--------+

pivot column: 4
pivot row: 1
 pivot: 1
1

|--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      1 |      0 |      0 |      1 |      0 |      0 |      0 |      1 |
|     20 |      1 |      0 |      0 |      1 |      0 |      0 |    100 |
|    200 |     20 |      1 |      0 |      0 |      1 |      0 |  10000 |
|--------+--------+--------+--------+--------+--------+--------+--------+
|   -100 |    -10 |      0 |      0 |      0 |     -1 |      1 | -10000 |
|--------+--------+--------+--------+--------+--------+--------+--------+
```

Note that in the first iteration had we made x_3 rather than x_1 enter the basis, we would have pivited directly to the final optimal tableau.

PROBLEMS REQUIRING AN UNUSUALLY LARGE NUMBER OF ITERATIONS

From a purist point of view, it would be even more reassuring to have a proof that, for *every* problem (4.1), the simplex method would require no more than, say, $10mn$ iterations to find an optimal solution. However, there is no such proof. Worse than that, there are examples of LP problems that make the simplex method go through an enormous number of iterations. V. Klee and G. J. Minty (1972) have shown that in the process of solving the problem

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} 10^{n-j} x_j & \\ \text{subject to} & \displaystyle\left(2 \sum_{j=1}^{i-1} 10^{i-j} x_j\right) + x_i \le 100^{i-1} & (i = 1, 2, \ldots, n) \\ & x_j \ge 0 & (j = 1, 2, \ldots, n) \end{array} \tag{4.2}$$

the simplex method goes through $2^n - 1$ iterations. (A proof is outlined in problems 4.2 and 4.3.) This number is quite frightening. For example, at the rate of 100 iterations per second (a reasonably generous estimate), problem (4.2) with $n = 50$ would take more than 300,000 years to solve! (The empirical and simulation results just quoted do *not* contradict this result. They simply suggest that problems requiring large numbers of iterations must be rare. For this reason, the Klee–Minty examples (4.2) and other similar examples are sometimes referred to as "pathological.")