

DM545/DM871

Linear and Integer Programming

Lecture 3

The Simplex Method

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Pool on Take-Home Tests

- Test 1: Friday, March 1
- Test 2: Tuesday, March 26

1. Simplex Method

- Standard Form

- Basic Feasible Solutions

- Algorithm

- Tableaux and Dictionaries

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A Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{i=1}^m a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

$$\max \quad [6 \ 8] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

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Standard Form

Every LP problem can be converted in the **standard form**:

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

$$\mathbf{c} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

- if equations, then put two constraints, $\mathbf{ax} \leq \mathbf{b}$ and $\mathbf{ax} \geq \mathbf{b}$
- if $\mathbf{ax} \geq \mathbf{b}$ then $-\mathbf{ax} \leq -\mathbf{b}$
- if $\min \mathbf{c}^T \mathbf{x}$ then $-\max(-\mathbf{c}^T \mathbf{x})$

and then be put in **equational standard form**:

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

1. "=" constraints
2. $\mathbf{x} \geq 0$ nonnegativity constraints
3. ($\mathbf{b} \geq 0$)
4. max

Transformation to Std Form

Every LP problem can be transformed in eq. std. form

1. introduce slack variables (or surplus)

$$5x_1 + 10x_2 + x_3 = 60$$

$$4x_1 + 4x_2 + x_4 = 40$$

2. if $x_1 \begin{matrix} \geq \\ \leq \end{matrix} 0$ then
$$\begin{aligned} x_1 &= x_1' - x_1'' \\ x_1' &\geq 0 \\ x_1'' &\geq 0 \end{aligned}$$

3. ($b \geq 0$)

4. $\min c^T x \equiv -\max(-c^T x)$

LP in $m \times n$ converted into LP with at most $(m + 2n)$ variables and m equations ($n \neq$ original variables, $m \neq$ constraints)

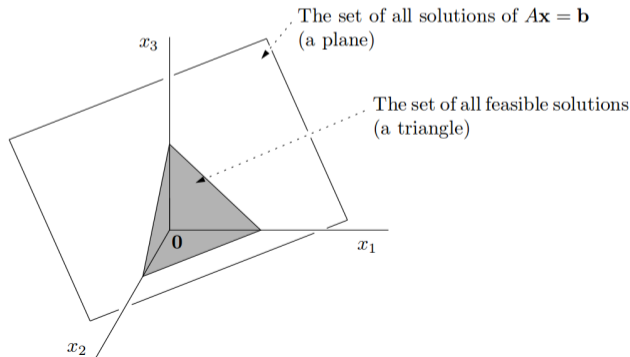
Geometry of LP in Eq. Std. Form

$$\max\{c^T x \mid Ax = b, x \geq 0\}$$

In \mathbb{R}^3 :

From linear algebra:

- the set of solutions of $Ax = b$ is an affine space (hyperplane not passing through the origin).
- $x \geq 0$ nonnegative orthant (octant in \mathbb{R}^3)



- $Ax = b$ is a system of equations that we can solve by Gaussian elimination
- Elementary row operations of $[A \mid b]$ do not affect set of feasible solutions
 - multiplying all entries in some row of $[A \mid b]$ by a nonzero real number λ
 - replacing the i th row of $[A \mid b]$ by the sum of the i th row and j th row for some $i \neq j$
- Let n' be the number of vars in eq. std. form.

we assume $n' \geq m$ and $\text{rank}([A \mid b]) = \text{rank}(A) = m$

ie, rows of A are linearly independent
otherwise, remove linear dependent rows

1. Simplex Method

Standard Form

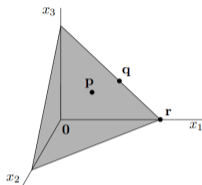
Basic Feasible Solutions

Algorithm

Tableaux and Dictionaries

Basic Feasible Solutions

Basic feasible solutions are the vertices of the feasible region:



More formally:

Let $B = \{1 \dots m\}$, $N = \{m + 1 \dots n + m = n'\}$ be subsets partitioning the columns of A : A_B be made of columns of A indexed by B :

Definition

$\mathbf{x} \in \mathbb{R}^n$ is a **basic feasible solution** of the linear program $\max\{\mathbf{c}^T \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ for an index set B if:

- $x_j = 0 \forall j \notin B$
- the square matrix A_B is nonsingular, ie, all columns indexed by B are lin. indep.
- $\mathbf{x}_B = A_B^{-1} \mathbf{b}$ is nonnegative, ie, $\mathbf{x}_B \geq 0$ (feasibility)

We call x_j for $j \in B$ **basic variables** and remaining variables **nonbasic variables**.

Theorem

A **basic feasible solution** is uniquely determined by the set B .

Proof:

$$Ax = A_B x_B + A_N x_N = b$$

$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

$$x_B = A_B^{-1} b$$

A_B is nonsingular hence one solution

Note: we call B a **(feasible) basis**

Definition

A **basic feasible solution** of a linear program with n variables is a feasible solution for which some n linearly independent constraints hold with equality.

Extreme points and basic feasible solutions are geometric and algebraic manifestations of the same concept:

Theorem

Let P be a (convex) polyhedron made of all feasible solutions of an LP in eq. std. form. For a point $v \in P$ the following are equivalent:

- (i) v is an extreme point (vertex) of P
- (ii) v is a basic feasible solution of LP

Proof: see text book [MG] sec. 4.4.

Theorem

Let $LP = \max\{\mathbf{c}^T \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ be feasible and bounded, then there is an optimal solution that is a basic feasible solution.

Proof. consequence of previous theorem and fundamental theorem of linear programming

A similar theorem is valid for arbitrary linear programs (not in eq. form)

However, an optimal solution does not need to be basic:

$$\max x_1 + x_2 \text{ subject to } x_1 + x_2 \leq 1$$

(unbounded case with infinite solutions)

- Idea for solution method:
- examine all basic solutions.
- There are finitely many: $\binom{m+n}{m}$.
- However, if $n = m$ then $\binom{2m}{m} \approx 4^m$.

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Simplex Method

$$\max \quad z = [6 \ 8] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Canonical eq. std. form: one decision variable is isolated in each constraint with coefficient 1 and does not appear in the other constraints nor in the obj. func. and b terms are positive

It gives immediately a basic feasible solution:

$$x_1 = 0, x_2 = 0, x_3 = 60, x_4 = 40$$

Is it optimal? Look at signs in $z \rightsquigarrow$ if positive then an increase would improve.

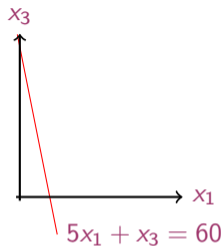
Let's try to increase a promising variable, ie, x_1 , one with positive coefficient in z

$$5x_1 + x_3 = 60$$

$$x_1 = \frac{60}{5} - \frac{x_3}{5}$$

$$x_3 = 60 - 5x_1 \geq 0$$

If $x_1 > 12$ then $x_3 < 0$

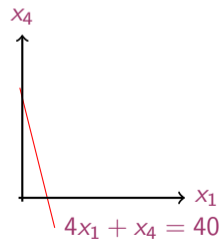


$$4x_1 + x_4 = 40$$

$$x_1 = \frac{40}{4} - \frac{x_4}{4}$$

$$x_4 = 40 - 4x_1 \geq 0$$

If $x_1 > 10$ then $x_4 < 0$



we can take the minimum of the two $\rightsquigarrow x_1$ increased to 10
 x_1 enters the basis and x_4 leaves it.

Dictionary Form

Hence, we increase x_1 to 10 and consequently set $x_4 = 0$
 That is, x_1 enters the basis and x_4 leaves it.

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 + x_3 = 60 \\ & 4x_1 + 4x_2 + x_4 = 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{array}{r} x_3 = 60 - 5x_1 - 10x_2 \\ x_4 = 40 - 4x_1 - 4x_2 \\ \hline z = \quad + 6x_1 + 8x_2 \end{array}$$

$$\begin{array}{r} x_3 = 10 - 5x_2 + 5/4x_4 \\ x_1 = 10 - x_2 - 1/4x_4 \\ \hline z = 60 + 2x_2 - 6/4x_4 \end{array}$$

Simplex Tableau

First simplex tableau:

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

we want to reach this new tableau

	x_1	x_2	x_3	x_4	$-z$	b
x_3	0	?	1	?	0	?
x_1	1	?	0	?	0	?
	0	?	0	?	1	?

Pivot operation:

1. Choose pivot:

column: one s with positive coefficient in obj. func.

row: ratio between coefficient b and pivot column: choose the one with smallest ratio:

$$\theta = \min_i \left\{ \frac{b_i}{a_{is}} : a_{is} > 0 \right\}, \quad \theta \text{ increase value of entering var.}$$

2. elementary row operations to update the tableau

- x_4 leaves the basis, x_1 enters the basis
 - Divide pivot row by pivot
 - Send to zero the coefficient in the pivot column of the first row
 - Send to zero the coefficient of the pivot column in the third (cost) row

	x_1	x_2	x_3	x_4	$-z$	b
I' = I - 5II'	0	5	1	-5/4	0	10
II' = II/4	1	1	0	1/4	0	10
III' = III - 6II'	0	2	0	-6/4	1	-60

From the last row we read: $2x_2 - 3/2x_4 - z = -60$, that is: $z = 60 + 2x_2 - 3/2x_4$.
 Since x_2 and x_4 are nonbasic we have $z = 60$ and $x_1 = 10, x_2 = 0, x_3 = 10, x_4 = 0$.

- Done? No! Let x_2 enter the basis

	x_1	x_2	x_3	x_4	$-z$	b
I' = I/5	0	1	1/5	-1/4	0	2
II' = II - I'	1	0	-1/5	1/2	0	8
III' = III - 2I'	0	0	-2/5	-1	1	-64

Definition (Reduced costs)

We call **reduced costs** the coefficients in the objective function of the nonbasic variables, \bar{c}_N

Proposition (Optimality Condition)

The basic feasible solution is **optimal** when the **reduced costs** in the corresponding simplex tableau are **nonpositive**, ie, such that:

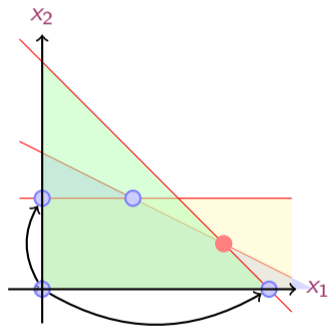
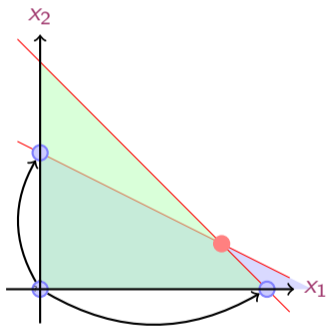
$$\bar{c}_N \leq 0$$

Proof: Let z_0 be the obj value when $\bar{c}_N \leq 0$.

For any other feasible solution $\tilde{\mathbf{x}} = [\tilde{\mathbf{x}}_B, \tilde{\mathbf{x}}_N]$, $\tilde{\mathbf{x}}_N \geq 0$ reachable from the tableau we have:

$$z = \mathbf{c}^T \tilde{\mathbf{x}} \quad \text{and from the last line of the tableau} \quad \mathbf{c}^T \tilde{\mathbf{x}} = z = z_0 + \bar{\mathbf{c}}_B^T \tilde{\mathbf{x}}_B + \bar{\mathbf{c}}_N^T \tilde{\mathbf{x}}_N \leq z_0$$

Graphical Representation



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Tableaux and Dictionaries

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

$$x_{n+i} = b_i - \sum_{j=1}^n a_{ij} x_j, \quad i = 1, \dots, m$$

$$z = \sum_{j=1}^n c_j x_j$$

Tableau

$$\left[\begin{array}{c|c|c|c} I & \bar{A}_N & 0 & \bar{b} \\ \hline 0 & \bar{c}_N & 1 & -\bar{d} \end{array} \right]$$

Dictionary

$$\begin{aligned} x_r &= \bar{b}_r - \sum_{s \notin B} \bar{a}_{rs} x_s, \quad r \in B \\ z &= \bar{d} + \sum_{s \notin B} \bar{c}_s x_s \end{aligned}$$

pivot operations in dictionary form:

choose col s with r.c. > 0

choose row with $\min\{-\bar{b}_i/\bar{a}_{is} \mid \bar{a}_{is} < 0, i = 1, \dots, m\}$

update: express entering variable and substitute in other rows

Example

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

$$x_3 = 60 - 5x_1 - 10x_2$$

$$x_4 = 40 - 4x_1 - 4x_2$$

$$z = \quad + 6x_1 + 8x_2$$

After 2 iterations:

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	$1/5$	$-1/4$	0	2
x_1	1	0	$-1/5$	$1/2$	0	8
	0	0	$-2/5$	-1	1	-64

$$x_2 = 2 - 1/5x_3 + 1/4x_4$$

$$x_1 = 8 + 1/5x_3 - 1/2x_4$$

$$z = 64 - 2/5x_3 - 1x_4$$

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